18.03 Problem Set 7, Fall 2017
(due in class on Thursday, Nov. 9)

Part I  (20 points)

EP = Edwards & Penney; SN = Supplementary Notes (all have solutions)
(Exercises listed as ‘others’ are some suggested choices for more practice.)

**Topic 18** (W, Nov. 1) Fundamental Matrix, matrix exponential.
Read: SN §§LS.5 and LS.6.1
Hand in: SN 4F/1a, 2abc.

**Topic 19** (R, Nov. 2) Variation of param., Exponential input, Euler’s method.
Read: Topic 19 notes.
Hand in: Part I problems 19.1ab, 19.2 (posted with the pset).

**Topic 20** (M, Nov. 6) Step and delta functions.
Read: Topic 20 notes.
Hand in: Part I problems 20.1-3 (posted with the pset).

**Continuation:** (T, Nov. 7) Discussion, review and catch up.

**Coming next**

**Topic 21** (W, Nov. 8) Fourier Series – basics.

**Topic 22** (R, Apr. 9) Fourier Series – continuation.
Read: Topic 22 notes.

Part II  (92 points)

Directions: Try each problem alone for 20 minutes. If you collaborate later, you must write up solutions independently. Consulting old problem sets is not permitted.

**Problem 1** (Topic 17) (20: 4,8,4,4)
DE·SYSTEMS is an avant-garde pas de deux expressing in the universal language of dance the cycles of attraction and disdain of a pair of star-crossed armadillos named Armand (A) and Babette (B). Let $x = x(t)$ represent the time-varying level of A’s attraction to B, and $y = y(t)$ represent the level of B’s attraction to A, both in some suitable units. (Negative values of $x$ or $y$ will mean negative attraction, i.e repulsion.) These being DE armadillos, we have, of course, equations giving the (coupled) rates of change of these levels of attraction, namely the $2 \times 2$ system of DE’s:

$$x' = x - y, \quad y' = 2x - y.$$ 

(a) Describe in words what this system of DE’s is saying about the feelings of A and B for each other.

Note: DE armadillos are known to be introspective, even to the point of brooding. (This is relevant to one of the two terms in each equation.)
(b) Take IC’s \( x(0) = 2, y(0) = 2 \). Solve the system using matrix methods. Express your answers in “amplitude-phase” form.

(c) Use the MIT Mathlet Vector Fields at [http://mathlets.org/mathlets/vector-fields/](http://mathlets.org/mathlets/vector-fields/) to obtain a graph of the solution to part (b) in the phase-plane – that is, the graph of \( y \) vs. \( x \) (the trajectory of the curve in the xy-plane). (You’ll need to pick the right system from the drop-down menu.)

Print out the phase plane picture and include it with your pset.

(d) Describe in words the unfortunate “minuet” A and B will be performing over time which is given numerically and graphically by the pair of solutions \( x = x(t), y = y(t) \) to this DE system.

**Problem 2** (Topic 19) (12: 4,5,2)

In this problem you will use the method of variation of parameters for inhomogeneous systems to find a particular solution to the ODE: \( x'' + \frac{1}{t} x' - \frac{1}{t^2} x = \delta(t - 1) \).

Two independent solutions to the homogeneous equation are \( x_1 = \frac{1}{t} \) and \( x_2 = t \).

(a) Let \( x' = y \) and convert the second order DE above into an inhomogeneous system. Also, write down a fundamental matrix for this system.

(b) Use variation of parameters to find a particular solution to the inhomogeneous system found in part(a). (Hint, integrals involving the delta function are easy.)

(c) Use your answer to part(b) to write down a particular solution to the original second order ODE.

**Problem 3** (Topic 19) (10: 5,5)

This problem has to do with the matrix: \( \Phi(t) = \begin{pmatrix} e^t & t e^{2t} \\ e^t & 2 t e^{2t} \end{pmatrix} \).

(a) Why can’t \( \Phi \) be the fundamental matrix of a constant coefficient \( 2 \times 2 \) system?

(b) Give a non-constant coefficient \( 2 \times 2 \) system, \( x' = A(t)x \), which has \( \Phi \) as a fundamental matrix.

**Problem 4** (Topic 20) (10)

Solve the initial value problem
\[
2x''' + 12x'' + 22x' + 12x = \delta(t) + e^t,
\]
with rest initial conditions.

Hint: the characteristic roots are small negative integers.

**Problem 5** (Topic 19) (20: 10,10)

(Continuing with the story of Armand and Babette.)

Armand and Babette, exhausted from their seemingly endless cycles of attraction and repulsion, decide nevertheless to give it one more try, and so are off to Armadillo couples therapy. The Therapist admits that, given their current emotional patterns (which will be impossible to change quickly), it doesn’t look good for a long-term stable happy relationship, but suggests a short-term external intervention to see if a break in the pattern
might give them some time to work on the deeper issues. The Therapist therefore sends them to the Wizard, who concocts a special potion for them. Again let $x(t)$ and $y(t)$ denote the time-varying levels of A’s attraction to B and B’s attraction to A respectively. The “interaction coefficients” in the rate DE’s for $x(t)$ and $y(t)$ are unchanged, since their emotional patterns are still the same; the effect of the Wizard’s intervention on the rates of change of their feelings for each other is then to add the “external” functions $f_1(t)$ and $f_2(t)$ respectively to these DE’s, so that we get

\[
x' = x - y + f_1(t) \quad y' = 2x - y + f_2(t).
\]

To ease your computational load, we’ll tell you that the corresponding homogeneous system has solution

\[
\begin{pmatrix} x \\ y \end{pmatrix} = c_1 \begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} \sin t \\ -\cos t + \sin t \end{pmatrix}.
\]

(a) Whatever the Wizard intended, the effect on A and B respectively of their scheduled ingestion of his potion turns out to be $f_1(t) = 3$ and $f_2(t) = -3$. So that apparently Armand has a good reaction and Babette a bad reaction to it.

Use our techniques for exponential input to find a particular solution to this inhomogeneous equation.

What is the effect of the potion on A and B’s situation? (Remember the DE’s describe their time-varying rates of affection.) In particular, explain how the apparent negative effect of the potion on B’s feelings for A could end up resulting in a net positive outcome for their relationship. (They don’t call him the Wizard for nothing.)

(b) For some reason (probably because these are DE armadillos), the Wizard decides to adjust the potion so that $f_1(t) = 3 + 3\cos 2t$ and $f_2(t) = -3 + 3\cos 2t$. Solve the resulting inhomogeneous DE. You should make use of superposition and your answer in part (b) to reduce the amount of computation needed.

---

**Problem 6**  (Topic 20)  (20: 5,5,5,5) **Lemmings really are adorable**

Back in your impulsive youth you helped a population of lemmings avoid extinction. But your methods led to some tedious differential equations that no one liked solving. Now that you’re older and wiser and truly understand impulses you are ready to help the lemmings again.

![Lemmings](http://www.birdphotos.com)
Recall the the population \( y \) of lemmings is modeled by
\[
y' + ky = f(t),
\]
where \( t \) is measured in year, \( k = 1.0 \) is the growthrate and \( f(t) \) is the input function.

(a) Your youthful input function could be described as a periodic box.

\[
f_h(t) = \begin{cases} 
\frac{1}{h} & \text{if } 0 < t < h \\
0 & \text{if } h < t < 1 \\
\frac{1}{h} & \text{if } 1 < t < 1 + h \\
0 & \text{if } 1 + h < t < 2 \\
\end{cases}
\]

(i) Graph this function for \( h = 1/2 \).

(ii) How many units of lemmings were added in each yearly cycle.

(b) One problem with the input in part (a) is that you had to spend half the year on the tundra. Another was that solving the DE with \( f_h(t) \) as input took about half a year and almost made you quit 18.03. So you decide to let \( h \) go to 0. That is, every year on January 1 you’ll bring a truckload of lemmings (1 truckload = 1 unit of lemmings) to the wildlife reserve and release them all at once.

Call the new input function \( f_I(t) \). Give its formula in terms of delta functions and sketch its graph. (Let \( t = 0 \) be January 1, 2017.)

(c) In your absence the lemming population dwindled to nothing. Solve the DE \( y' + ky = f_I(t) \) with rest initial conditions.

(d) Now we’ll look at this graphically using a mathlet. Open http://mathlets.org/mathlets/periodic-box/

As usual, start the applet and familiarize yourself with its controls. Set \( k = 1 \) and \( h = 0.5 \).

(i) What happens to the response from rest as \( h \) goes to 0?

(ii) What happens to the impulse train response as \( t \) gets large.

End of problem set 7.