1. Measurement of Angular Momentum

Let us recall that
\[
\hat{L}_+ |\ell, m\rangle = \hbar \sqrt{\ell (\ell + 1) - m(m + 1)} |\ell, m + 1\rangle,
\]
\[
\hat{L}_- |\ell, m\rangle = \hbar \sqrt{\ell (\ell + 1) - m(m - 1)} |\ell, m - 1\rangle,
\]
and
\[
\hat{L}_x = \frac{\hat{L}_+ + \hat{L}_-}{2}, \quad \hat{L}_y = \frac{\hat{L}_+ - \hat{L}_-}{2i}.
\]  

(a) (2 points) First of all, \(\hat{L}^2 = \hbar^2 \ell (\ell + 1) \hat{1} = 2\hbar^2 \hat{1}\). Then
\[
\hat{L}_z = \hbar \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix},
\]
and using the formulas (1)(2) we find
\[
\hat{L}_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix}, \quad \hat{L}_y = \frac{\hbar}{\sqrt{2i}} \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 1 \\
0 & -1 & 0
\end{pmatrix}.
\]

(b) (2 points)
\[
\hat{L}_y |\ell = 1, m_y = 1\rangle = \hbar |\ell = 1, m_y = 1\rangle \Rightarrow |\ell = 1, m_y = 1\rangle = \frac{1}{2} \left( |1, 1\rangle + i \sqrt{2} |1, 0\rangle - |1, -1\rangle \right).
\]

(c) (2 points)
\[
\hat{L}^2 |\ell = 1, m_y = 1\rangle = 2\hbar^2 |\ell = 1, m_y = 1\rangle,
\]
so there is only one possible result, \(2\hbar^2\), with 100% probability.

(d) (2 points) From (b), we see that the possible results are \(\pm \hbar\), 0, and the associated probabilities are
\[
P(\pm \hbar) = |\langle 1, 1\rvert 1, m_y = 1\rangle|^2 = \frac{1}{4}, \quad P(0) = \frac{1}{2}, \quad P(-\hbar) = \frac{1}{4}.
\]

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(e) (2 points) The possible outcomes are still $\pm \hbar$ and 0. The probabilities are given by

$$
|\langle \ell = 1, m_y = \pm 1 | 1, 1 \rangle|^2, \quad |\langle \ell = 1, m_y = 0 | 1, 1 \rangle|^2.
$$

Instead of calculating them directly, we can use the following argument. Let us call $U$ the unitary operator performing a counterclockwise rotation by 90 degrees around the $\hat{x}$ axis. It satisfies

$$
U^\dagger L_y U = L_z, \quad U^\dagger L_z U = -L_y.
$$

Using the above properties we can derive that $U$ maps eigenstates of $L_z$ to eigenstates of $L_y$ with the same eigenvalue and vice versa $U$ maps eigenstates of $L_y$ to eigenstates of $L_z$ with the opposite eigenvalues. In fact

$$
L_y U |\ell, m_z \rangle = U U^\dagger L_y U |\ell, m_z \rangle = \hbar m_z U |\ell, m_z \rangle,
$$

$$
L_z U |\ell, m_y \rangle = U U^\dagger L_z U |\ell, m_y \rangle = -\hbar m_y U |\ell, m_y \rangle.
$$

Now, let us go back to the probabilities we have to calculate. We find

$$
P_y(m_y) = |\langle \ell = 1, m_y | \ell = 1, m_z = 1 \rangle|^2 = |\langle \ell = 1, m_y | U^\dagger U | \ell = 1, m_z = 1 \rangle|^2
$$

$$
= |\langle \ell = 1, m_z = -m_y | \ell = 1, m_y = 1 \rangle|^2 = P_z(-m_y),
$$

where $P_z(m_z)$ are the probabilities that a measurement of $L_z$ on a state $|\ell = 1, m_y = 1 \rangle$ yields $\hbar m_z$, which we calculated in (d). Therefore, we can just take the result from (d) and write

$$
P_y(1) = \frac{1}{4}, \quad P_y(0) = \frac{1}{2}, \quad P_y(-1) = \frac{1}{4}.
$$

2. Particle on a Sphere

(a) (4 points) Since

$$
\frac{\rho^2}{2M} = -\frac{\hbar^2}{2M} \nabla^2 = \frac{\rho^2}{2M} + \frac{L^2}{2Mr^2}
$$

we find

$$
\frac{L^2}{2Mr^2} = -\frac{\hbar^2}{2Mr^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right].
$$

The Schrödinger equation

$$
-\frac{\hbar^2}{2Mr^2} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right] \psi(\theta, \varphi) = E \psi(\theta, \varphi),
$$
is solved by the spherical harmonics $Y_{\ell,m}$ with eigenvalue $E_{\ell} = \frac{\hbar^2}{2\mu} \ell (\ell + 1)$, $\ell = 0, 1, \ldots$. The energy level $E_{\ell}$ has degeneracy $2\ell + 1$. Because of this degeneracy, the Hamiltonian does not constitute a complete set of commuting variables by itself. Instead $\{H, \hat{L}_z\}$ represents a complete set of commuting variables.

(b) (3 points) The probability is given by

$$
\int_{\theta=\pi/4}^{\theta=\pi/2} \int_{\varphi=0}^{\varphi=2\pi} |\langle \vec{r}| \psi \rangle|^2 d\Omega = \frac{1}{8\pi} \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{\varphi=0}^{\varphi=2\pi} \left(1 + 3 \cos^2 \varphi \sin^2 \theta + 2\sqrt{3} \cos \varphi \sin \theta\right) \sin \theta d\theta d\varphi
$$

$$
= \frac{1}{8\pi} \int_{\theta=\pi/4}^{\theta=\pi/2} \left(2\pi + 3\pi \sin^2 \theta\right) \sin \theta d\theta = \frac{9\sqrt{2}}{32} \approx 0.398.
$$

(c) (3 points) To answer the question, we are going to express $\psi(\theta, \varphi)$ as a linear combination of spherical harmonics. Using

$$
Y_{0,0} = \frac{1}{\sqrt{4\pi}}, \quad Y_{1,1} = -\sqrt{\frac{3}{8\pi}} \sin \theta (\cos \varphi + i \sin \varphi), \quad Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \sin \theta (\cos \varphi - i \sin \varphi),
$$

we find that

$$
\psi(\theta, \varphi) = \frac{1}{\sqrt{2}} Y_{0,0} + \frac{1}{2} Y_{1,-1} - \frac{1}{2} Y_{1,1}.
$$

Then, the possible results for the energy are $0, \pm \frac{\hbar^2}{2m}$ with probability $1/2, 1/4, 1/4$ respectively. The possible results for $\hat{L}^2$ are $0, \pm \hbar^2$ with probability $1/2, 1/4, 1/4$. Finally the possible results for $\hat{L}_z$ are $0, \pm \hbar$ with probability $1/2, 1/4, 1/4$ again.

3. “Cartesian Harmonics”

(a) (5 points) $Y_{0,0} = \frac{1}{\sqrt{4\pi}}$ corresponding to $a = b = c = 0$.

$$
Y_{1,0} = \sqrt{\frac{3}{4\pi}} \cos \theta = \sqrt{\frac{3}{4\pi}} \frac{x}{r},
$$

$$
Y_{1,1} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\varphi} = -\sqrt{\frac{3}{8\pi}} \frac{x + iy}{r}, \quad Y_{1,-1} = \sqrt{\frac{3}{8\pi}} \sin \theta e^{-i\varphi} = \sqrt{\frac{3}{8\pi}} \frac{x - iy}{r},
$$

$$
Y_{2,0} = \sqrt{\frac{5}{16\pi}} (3 \cos^2 \varphi - 1) = \sqrt{\frac{5}{16\pi}} \left(\frac{x^2}{r^2} - \frac{x^2 + y^2}{r^2}\right),
$$

$$
Y_{2,\pm1} = \mp \sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{\pm i\varphi} = \mp \sqrt{\frac{15}{8\pi}} \frac{x \pm iy}{r},
$$

$$
Y_{2,\pm2} = \sqrt{\frac{15}{32\pi}} \left(\sin \theta e^{\pm i\varphi}\right)^2 = \sqrt{\frac{15}{32\pi}} \left(\frac{x \pm iy}{r}\right)^2.
$$
(b) (5 points) From (a), we see that the angular part of the wavefunction corresponds to spherical harmonics with $\ell = 2$ only. Therefore the probability that a measurement of $\hat{L}^2$ yields 0 is zero. Vice versa, the probability that a measurement of $\hat{L}^2$ yields $6\hbar^2$ is 1. To answer the last question we need to express the angular part of the wavefunction as a linear combination of $\ell = 2$ spherical harmonics. Note that

$$xz = \frac{r^2}{2} \sqrt{\frac{8\pi}{15}} (Y_{2,-1} - Y_{2,1}) , \quad yz = -\frac{r^2}{2i} \sqrt{\frac{8\pi}{15}} (Y_{2,-1} + Y_{2,1}) ,$$

$$xy = \frac{r^2}{2i} \sqrt{\frac{8\pi}{15}} (Y_{2,2} - Y_{2,-2}) .$$

Then we can rewrite the wavefunction as

$$\psi(x, y, z) = Cr^2 \sqrt{\frac{8\pi}{15}} \left( \frac{1}{2i} Y_{2,2} - \frac{1}{2i} Y_{2,-2} - \frac{i + 1}{2i} Y_{2,1} + \frac{i - 1}{2i} Y_{2,-1} \right) e^{-ar^2} .$$

Since $Y_{2,0}$ is not there, the probability to measure $m = 0$ is vanishing. The relative probabilities to yield $m = 2, 1, -1, -2$ are proportional to the square moduli of the coefficients of the relative spherical harmonics, namely $1/4, 1/2, 1/2, 1/4$. Therefore the probabilities to measure $m = 2, 1, -1, -2$ are $1/6, 1/3, 1/3, 1/6$.

4. Rotation Operator

$$e^{-it\cdot \hat{r} \cdot \varphi_0 / \hbar} f(\theta, \varphi) = e^{-\varphi_0 \frac{\partial}{\partial \varphi}} f(\theta, \varphi)$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (-\varphi_0)^n \frac{\partial^n}{\partial \varphi^n} f(\theta, \varphi) ,$$

$$= f(\theta, \varphi) - \varphi_0 \frac{\partial}{\partial \varphi} f(\theta, \varphi) + \frac{1}{2!} (-\varphi_0)^2 \frac{\partial^2}{\partial \varphi^2} f(\theta, \varphi) + \ldots = f(\theta, \varphi - \varphi_0) .$$

5. Half Integer Spherical Harmonics Are Not Allowed

$$\hat{L}_x = y \hat{p}_z - z \hat{p}_y = -i\hbar \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right) = i\hbar \left( \sin \varphi \frac{\partial}{\partial \theta} + \cot \theta \cos \varphi \frac{\partial}{\partial \varphi} \right) ,$$

$$\hat{L}_y = z \hat{p}_x - x \hat{p}_z = -i\hbar \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right) = i\hbar \left( -\cos \varphi \frac{\partial}{\partial \theta} + \cot \theta \sin \varphi \frac{\partial}{\partial \varphi} \right) \Rightarrow$$

$$\hat{L}_\pm = \hat{L}_x \pm i\hat{L}_y = \pm \hbar e^{\pm i\varphi} \left( \frac{\partial}{\partial \theta} \pm i \cot \theta \frac{\partial}{\partial \varphi} \right) .$$
(a) (5 points) Then

\[
\hat{L}_+ Y_{1/2,1/2} = \hbar e^{i\varphi} \left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right) c \sqrt{\sin \theta} e^{i\varphi/2} = \hbar c e^{i3\varphi/2} \left( \frac{\cos \theta}{2 \sqrt{\sin \theta}} - \frac{1}{2} \cot \theta \sqrt{\sin \theta} \right) = 0 ,
\]

\[
\hat{L}_- Y_{1/2,-1/2} = -\hbar e^{-i\varphi} \left( \frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \varphi} \right) c \sqrt{\sin \theta} e^{-i\varphi/2}
= -\hbar c e^{-i3\varphi/2} \left( \frac{\cos \theta}{2 \sqrt{\sin \theta}} - \frac{1}{2} \cot \theta \sqrt{\sin \theta} \right) = 0 ,
\]

\[
\hat{L}_- Y_{1/2,1/2} = -\hbar e^{-i\varphi} \left( \frac{\partial}{\partial \theta} - i \cot \theta \frac{\partial}{\partial \varphi} \right) c \sqrt{\sin \theta} e^{i\varphi/2} = -\hbar c e^{-i\varphi/2} \left( \frac{\cos \theta}{2 \sqrt{\sin \theta}} + \frac{1}{2} \cot \theta \sqrt{\sin \theta} \right)
= -\hbar c e^{-i\varphi/2} \cos \theta \sqrt{\sin \theta} \neq Y_{1/2,-1/2} .
\]

(b) (5 points)

\[
\hat{L}_+ Y_{3/2,3/2} = \hbar e^{i\varphi} \left( \frac{\partial}{\partial \theta} + i \cot \theta \frac{\partial}{\partial \varphi} \right) c (\sin \theta)^{3/2} e^{i3\varphi/2}
= \hbar c e^{i3\varphi/2} \left( \frac{3}{2} \sqrt{\sin \theta} \cos \theta - \frac{3}{2} \cot \theta (\sin \theta)^{3/2} \right) = 0 .
\]

Next

\[ Y_{3/2,-3/2} \propto \hat{L}_+^2 Y_{3/2,3/2} , \]

which yields

\[ Y_{3/2,-3/2} \propto e^{-3i\varphi/2} \frac{\cos \theta (\cos 2\theta - 2)}{(\sin \theta)^{3/2}} . \]

Then

\[ \hat{L}_- Y_{3/2,-3/2} \propto e^{-3i\varphi/2} \frac{\sin \theta (\cos \theta)^2}{(\sin \theta)^{5/2}} \neq 0 , \]

and

\[ \left| Y_{3/2,-3/2} \right|^2 \propto e^{-(2\theta - 2)^2} \frac{(\cos \theta)^2}{(\sin \theta)^3} , \]

which has a pole of order 3 at \( \theta = 0 \) and \( \theta = \pi \), which makes this function non-normalizable.

(c) (5 points)

\[ [q_1, q_2] = \frac{1}{2} \left( [x, x] + \frac{a^4}{\hbar^2} [p_y, p_y] + \frac{a^2}{\hbar} [p_y, x] - \frac{a^2}{\hbar} [x, p_y] \right) = 0 , \]

\[ [p_1, p_2] = \frac{1}{2} \left( [p_x, p_x] + \frac{\hbar^2}{a^4} [y, y] - \frac{\hbar}{a^2} [y, p_x] + \frac{\hbar}{a^2} [p_x, y] \right) = 0 , \]

\[ [q_1, p_1] = \frac{1}{2} ([x, p_x] - [p_y, y]) = i\hbar , \quad [q_2, p_2] = \frac{1}{2} ([x, p_x] - [p_y, y]) = i\hbar , \]

5
Inverting the expressions for \( q_1, q_2, p_1, p_2 \) we find

\[
x = \frac{q_1 + q_2}{\sqrt{2}}, \quad p_x = \frac{1}{\sqrt{2}}(p_1 + p_2), \quad y = \frac{a^2}{\hbar \sqrt{2}}(p_2 - p_1), \quad p_y = \frac{\hbar}{a^2 \sqrt{2}}(q_1 - q_2),
\]

then

\[
L_z = xp_y - yp_x = \frac{\hbar}{2a^2}(q_1 + q_2)(q_1 - q_2) - \frac{a^2}{2\hbar}(p_2 - p_1)(p_1 + p_2)
\]

\[
= \frac{\hbar}{2a^2}(q_1^2 - q_2^2) + \frac{a^2}{2\hbar}(p_1^2 - p_2^2) = \left( \frac{p_1^2}{2m} + \frac{1}{2}m\omega^2q_1^2 \right) - \left( \frac{p_2^2}{2m} + \frac{1}{2}m\omega^2q_2^2 \right) = H_1 - H_2,
\]

where

\[
m = \frac{\hbar}{a^2}, \quad \omega = 1.
\]

Finally, since the eigenvalues of \( H_1 \) and \( H_2 \) are given by \( \hbar(n_1 + 1/2), \hbar(n_2 + 1/2) \), \( n_1, n_2 \in \mathbb{N} \) we conclude that the eigenvalues of \( L_z \) are

\[
\hbar(n_1 - n_2),
\]

which are integers.

6. Solving the Schrödinger Equation for the Hydrogen Atom

(a) (1 point) Upon setting \( r = bx \), the Schrödinger equation reads

\[
\left[ -\frac{\hbar^2}{2\mu b^2} \frac{d^2}{dx^2} + \frac{\hbar^2\ell(\ell + 1)}{2\mu b^2 x^2} - \frac{Ze^2}{bx} \right] u(x) = -Bu(x),
\]

and multiplying everything by \( 2\mu b^2/\hbar^2 \) we get

\[
\left[ -\frac{d^2}{dx^2} + \frac{\ell(\ell + 1)}{x^2} - \frac{2\mu bZe^2}{\hbar^2 x} \right] u(x) = -\frac{2\mu b^2B}{\hbar^2} u(x).
\]

So we set

\[
b = \frac{\hbar^2}{2\mu Ze^2} = \frac{a_0}{2Z},
\]

and we find

\[
\kappa^2 = \frac{2\mu b^2B}{\hbar^2} = \frac{\hbar^2 B}{2\mu(Ze^2)^2}.
\]

(b) (1 point) For small \( x \), the differential equation can be approximated to

\[
\left[ -\frac{d^2}{dx^2} + \frac{\ell(\ell + 1)}{x^2} \right] u(x) = -\kappa^2 u(x).
\]
which admits the following solutions
\[ u_1(x) = x^{\ell+1}, \quad u_2(x) = \frac{1}{x^{\ell}}. \]

Since the second solution has a pole at \( x = 0 \), we discard it and we find that \( u(x) \) goes like \( x^{\ell+1} \). Conversely, for large \( x \), the differential equation can be approximated to
\[
\frac{d^2}{dx^2}u(x) = \kappa^2 u(x),
\]
which is solved by \( e^{\pm \kappa x} \). Taking \( \kappa > 0 \), the allowed asymptotic behaviour is \( u(x) \sim e^{-\kappa x} \).

(c) (2 points) Plugging \( x^{\ell+1}e^{-\kappa x} \) into the equation we find
\[
\left[-(\ell(\ell+1)x^{\ell-1} + 2(\ell+1)(-\kappa)x^{\ell} + \kappa^2 x^{\ell+1}) + \ell(\ell+1)x^{\ell-1} - x^{\ell}\right]e^{-\kappa x} = -\kappa^2 x^{\ell+1}e^{-\kappa x}
\]
which is equivalent to
\[
(2(\ell + 1)\kappa - 1)) x^{\ell} e^{-\kappa x} = 0 \quad \Rightarrow \quad \kappa = \frac{1}{2(\ell + 1)}.
\]

(d) (1 point) From (a), we find
\[
\kappa^2 = \frac{1}{4(\ell + 1)^2} = \frac{\hbar^2 B}{2\mu(Ze^2)^2} \Rightarrow B = \frac{\mu(Ze^2)^2}{2\hbar^2(\ell + 1)^2} = 13.6 \frac{Z^2}{(\ell + 1)^2} \text{ eV}
\]

(e) (1 point) The wavefunction \( u(x) \) has no nodes for \( x > 0 \), therefore it must be the lowest energy state for given \( \ell \).

(f) (2 points)
\[
\frac{d^2}{dx^2}u(x) = Ne^{-\kappa x} \left[ \kappa^2 x^{\ell+1}(1 + cx) - 2\kappa((\ell + 1)x^{\ell} + c(\ell + 2)x^{\ell+1}) + \ell(\ell + 1)x^{\ell-1} + c(\ell + 1)(\ell + 2)x^{\ell}\right],
\]
the differential equation is equivalent to
\[
2\kappa((\ell + 1)x^{\ell} + c(\ell + 2)x^{\ell+1}) - \ell(\ell + 1)x^{\ell-1} - c(\ell + 1)(\ell + 2)x^{\ell} + \ell(\ell + 1)x^{\ell-1}(1 + cx) - x^{\ell}(1 + cx) = 0,
\]
which gives
\[
x^{\ell+1} [2\kappa c(\ell + 2) - c] + x^{\ell} [2\kappa(\ell + 1) - c(\ell + 1)(\ell + 2) + c \ell(\ell + 1) - 1] = 0.
\]
The two factors multiplying \( x^{\ell+1} \) and \( x^{\ell} \) have to vanish simultaneously. This yields
\[
\kappa = \frac{1}{2(\ell + 2)}, \quad c = -\frac{1}{2(\ell + 1)(\ell + 2)}.
\]
and the binding energy is
\[ B = \frac{\mu (Ze^2)^2}{2 \hbar^2 (\ell + 1)^2} = 13.6 \frac{Z^2}{(\ell + 2)^2} eV < 13.6 \frac{Z^2}{(\ell + 1)^2} eV. \]

(g) (2 points) For \( n = 1, \nu = 0 = \ell = m \) and
\[ \psi(r, \theta, \varphi) = \frac{1}{\sqrt{\pi a_0^3}} e^{-\frac{r}{a_0}}, \quad B = 13.6 \text{ eV}. \]

Let us take \( n = 2 \). We have the following cases.

- \( \nu = 1, \ell = m = 0 \)
\[ \psi(r, \theta, \varphi) = \frac{1}{\sqrt{8\pi a_0^3}} \left(1 - \frac{r}{2a_0}\right) e^{-\frac{r}{2a_0}}, \quad B = \frac{13.6}{4} \text{ eV}, \]

- \( \nu = 0, \ell = m = 1 \)
\[ \psi(r, \theta, \varphi) = \frac{1}{8\sqrt{\pi a_0^3}} r \sin \theta e^{i\varphi} e^{-\frac{r}{2a_0}}, \quad B = \frac{13.6}{4} \text{ eV}, \]

- \( \nu = 0, \ell = 1, m = 0 \)
\[ \psi(r, \theta, \varphi) = \frac{1}{4\sqrt{2\pi a_0^3}} r \cos \theta e^{-\frac{r}{2a_0}}, \quad B = \frac{13.6}{4} \text{ eV}, \]

- \( \nu = 0, \ell = 1, m = -1 \)
\[ \psi(r, \theta, \varphi) = \frac{1}{8\sqrt{\pi a_0^3}} r \sin \theta e^{-i\varphi} e^{-\frac{r}{2a_0}}, \quad B = \frac{13.6}{4} \text{ eV}. \]

Let us explain the meaning of the “angular plots”. For each value of the angles \((\theta, \varphi)\) you find a point on the surfaces, which also defines a three-dimensional vector, going from the origin of the three coordinate system to the point on the surface. The length of this vector is proportional to \(|Y_{\ell,m}|^2\). Basically, the plotted surfaces have the following parametrization
\[ \{x = |Y_{\ell,m}|^2(\theta, \varphi) \sin \theta \cos \varphi, y = |Y_{\ell,m}|^2(\theta, \varphi) \sin \theta \sin \varphi, z = |Y_{\ell,m}|^2(\theta, \varphi) \cos \theta\}. \]
Figure 1. A plot of $|Y_{0,0}|^2$, relative to the cases $n = 1, \ell = 0$, $n = 2, \ell = 0$.

Figure 2. A plot of $|Y_{1,\pm 1}|^2$ relative to the cases $n = 2, \ell = 1, m = \pm 1$. 
Figure 3. A plot of $|Y_{1,0}|^2$ relative to the case $n = 2, \ell = 1, m = 0$. 
Figure 4. Radial probability density for $n = 1, \ell = 0$; the radius is measured in units of $a_0$.

Figure 5. Radial probability density relative to $n = 2, \ell = 0$.

Figure 6. Radial probability density relative to $n = 2, \ell = 1$