Midterm Exam

Instructions: Carefully read these instructions! Failure to follow them may lead to deductions from your grade.

1. Immediately write your name at the bottom of this page and of the last page of the exam.

2. You will turn in only the exam, so write all answers on the exam.

3. Do not in any way communicate with other people taking the exam. Any communication – even if it is not about the exam – will result in a grade of zero.

4. Do not use computers during the exam. Do not use notes or books other than the ‘cheat sheet’ permitted by me.

5. There are 5 problems and you have 80 minutes to complete the test.

Last name:_________________________ First name: ________________

Section (circle one): E (1-2:30) F (2:30-4)
**Question 1** [20 points]

As a president of 401.com you need to choose between two investment projects. The first project, project A, requires an initial investment of $100,000 and will generate $15,000 per year for the next ten years. The first cash flow will arrive exactly in one year from now. The second project, project B, lasts forever and requires annual investments which start at $20,000 and decline at 3% per year. On the plus side, it generates constant annual cash flows of $15,000. Investments must be made in the beginning of every year, i.e., the first investment is due at time 0. The cash flows arrive at the end of every year, i.e., the first cash flow arrives at time 1. The term structure of spot interest rates is flat at 5%. (Here and in the rest of the problems interest rates are quoted as EAR).

a. (10 points) Compute the present value of project A.

b. (10 points) Compute the present value of project B.

**Solution**

a. Using the annuity formula,

\[ PV(A) = -100,000 + 15,000 \times \left( \frac{1}{0.05} - \frac{1}{0.05 \times 1.05^{10}} \right) = 15,826 \]

b. Using the perpetuity formula,

\[ PV(\text{Cash Flows}) = \frac{15,000}{0.05} = 300,000 \]

By the growing perpetuity formula, with the growth rate \( g = -3\% \),

\[ PV(\text{Investments}) = 20,000 + \frac{20,000 \times (1 - 0.03)}{0.05 - (-0.03)} = 262,500 \]

Thus,

\[ PV(B) = PV(\text{Cash Flows}) - PV(\text{Investments}) = 300,000 - 262,500 = 37,500. \]

**Question 2** [20 points]

The term structure today is flat at 5%. After one year, the term structure happens to move up to 6%, while still staying flat.

a. (5 points) What is the return between today and next year, of a zero-coupon bond with maturity of one year?
b. (5 points) What is the return between today and next year, of a zero-coupon bond with maturity of two years?

c. (5 points) What is the return between today and two years from now of a 7% annual coupon bond with a face value of $100 and maturity of two years? (Remember that coupons must be re-invested at the risk-free rate).

d. (5 points) If the interest rates did not change and remained at 5%, what would be the return between today and one year from now of a 7% annual coupon bond with maturity of two years? (Note that this one-year period covers the first coupon payment).

Solution

a. Since it’s one-year bond, its return is not affected by the change in interest rates, and equals 5%.

b. The price of the bond at time 0 is

\[ P_0 = \frac{100}{1.05^2} = 90.70 \]

The price at time 1 becomes

\[ P_1 = \frac{100}{1.06} = 94.34 \]

The return is then equal to

\[ R = \frac{94.34}{90.70} - 1 = 4.01\% \]

c. The total amount of money the bond generates at time 2 is

\[ 107 + 7 \times 1.06 = 114.42 \]

Its price at time 0 were

\[ P_0 = \frac{7}{1.05} + \frac{107}{1.05^2} = 103.72 \]

Thus the total return is

\[ R = \frac{114.42}{103.72} - 1 = 10.32\% \]

or, expressed in annual terms, \(1.1032^{1/2} - 1 = 5.03\%\).

Question 3 [15 points]

You are given the following information:
<table>
<thead>
<tr>
<th>Bond</th>
<th>Coupon Rate</th>
<th>Maturity</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>0%</td>
<td>2</td>
<td>90.70</td>
</tr>
<tr>
<td>B</td>
<td>5%</td>
<td>2</td>
<td>100.05</td>
</tr>
<tr>
<td>C</td>
<td>10%</td>
<td>3</td>
<td>111.07</td>
</tr>
</tbody>
</table>

All coupon payments are annual and par values are 100.

a. (5 points) Determine 1-, 2-, and 3-year spot rates based on the above information.

b. (5 points) Compute the annual forward rate from year one to year two, i.e., $1f_1$.

c. (5 points) A 3-year 5% coupon bond with face value $100 is trading at $102.00. Show that this bond is mispriced. Describe in detail the trading strategy you would follow to benefit from the mispricing, i.e., report your exact trades.

Solution

a. From the price of A, the two-year spot rate is

$$r_2 = \left(\frac{100}{90.7}\right)^{1/2} - 1 = 5\%$$

Because

$$P_B = \frac{5}{1 + r_1} + \frac{105}{1.05^2} = 100.05$$

we find that

$$\frac{5}{1 + r_1} = 4.81$$

and therefore $r_1 = 4\%$. Now, from

$$P_C = \frac{10}{1.04} + \frac{10}{1.05^2} + \frac{110}{(1 + r_3)^3} = 111.07$$

we find

$$\frac{110}{(1 + r_3)^3} = 92.38$$

and hence $r_3 = 6\%$.

b.

$$1 + 1f_1 = \frac{(1 + r_2)^2}{1 + r_1} = \frac{1.05^2}{1.04} = 1.0601$$

so $1f_1 = 6.01\%$. 
c. We use absence of arbitrage to figure out the theoretical price of the bond. The replicating portfolio, \((x_1, x_2, x_3)\) must satisfy

\[
\begin{align*}
0x_1 + 5x_2 + 10x_3 &= 5 \\
100x_1 + 105x_2 + 10x_3 &= 5 \\
0x_1 + 0x_2 + 110x_3 &= 105
\end{align*}
\]

We find that 
\[x_3 = \frac{105}{110} = 0.9545, \quad x_2 = \frac{(5 - 10 \times 0.9545)}{5} = -0.9090, \quad \text{and} \quad x_1 = \frac{(5 - 105 \times (-0.9090) - 10 \times 0.9545)}{100} = 0.9090.\]

Then the theoretical price of the new bond is 
\[90.70x_1 + 100.05x_2 + 111.07x_3 = 97.52.\]

To construct and arbitrage trade, we short sell the new bond, buy 0.9545 units of bond C, short sell 0.9090 units of bond B and buy 0.9090 units of bond A.

**Question 4 [25 points]**

Multiple choice questions. Please circle your answer, provide a one-line explanation.

a. (6 points) Holding the one-year real interest rate constant, if the nominal one-year interest rate where to increase by 1%, it would imply that the inflation rate over the same period

(a) Increased.
(b) Declined.
(c) Stayed the same.
(d) It can go either way, impossible to tell from the provided data.

b. (6 points) Consider two treasury bonds, A and B. Both have 5 years to maturity, A pays a 5% coupon rate, B pays a 7% coupon rate. Which of bonds A and B has higher modified duration,

(a) A.
(b) B.
(c) The same for A and B.
(d) It can go either way, impossible to tell from the provided data.

c. (6 points) A ten-year bond with a coupon rate of 6% and a face value of $100 is priced at $98. Let the yield to maturity be denoted by \(y\). Which of the following statements is true:

(a) \(y > 6\%\).
(b) \(y < 6\%\).
(c) \(y = 6\%\).
(d) It can go either way, impossible to tell from the provided data.
d. (4 points, this may require a calculation) Suppose the one-year spot rate $r_1 = 5\%$ and the two-year rate $r_2 = 6\%$. At time 0 you enter into a forward contract to buy, in exactly one year from now, a one-year zero-coupon bond. Suppose that in one year from now the term structure of interest rates changes, so that a one year rate becomes 6%. Will you experience a profit or a loss on your forward contract?

   (a) Profit.
   (b) Loss.
   (c) No effect.
   (d) It can go either way, impossible to tell from the provided data.

e. (3 points, a little harder) You are a trader for an investment bank in charge of a portfolio of bonds. At 10:00 am this morning you computed the modified duration of your portfolio to be $D^* = 6$ years and convexity equal to $\Gamma = 120$. At 10:30 am this morning, a news of unexpectedly high GDP growth came out, and the interest rates increased by .5%. Assume that the term structure of interest rates was flat both before and after the news. After the news, the modified duration of your portfolio will be

   (a) $D^* > 6$.
   (b) $D^* < 6$.
   (c) $D^* = 6$.
   (d) It can go either way, impossible to tell from the provided data.

Solution

a. (a) Nominal rate and inflation are positively related,

\[ 1 + r_{\text{real}} = \frac{1 + r_{\text{nominal}}}{1 + \text{inflation}} \]

b. (a) Duration declines with the coupon rate.

c. (a) When YTM exceeds the coupon rate, the bond trades below the par value.

d. (a) Since

\[ 1 + f = \frac{1.06^2}{1.05} \]

it must be that $f > 6\%$. Since the realized rate is below the forward rate, we were able to buy the bond cheaper than its market price at time 1, so the forward produced a profit.

e. (b) Because $\Gamma > 0$, the performance profile is convex and effective duration declines with the interest rates.
Question 5 [20 points]

An investment bank just sold an exotic fixed income security XYZ to a client. This security pays $100 in the first years of its life, and then its payments increase at 2% per year, i.e., the cash flow at the end of the second year is $102. Payments are due at the end of every year, starting in one year from today, and extend forward indefinitely. The term structure of interest rates is flat at 6%.

a. (6 points) Assuming that the security XYZ was fairly priced, how much money did the bank receive for it?

b. (6 points) Maintain the assumption that the term structure of spot interest rates is flat, denote the interest rate by $r$. Assuming that $r > 2\%$, derive a general formula for the price of XYZ.

c. (4 points, a little harder) If the interest rate suddenly changes by $\Delta r$, what will be the approximate change in the price of XYZ? (Hint: use the formula from the previous part and the fact that $\Delta f(x) \approx \frac{df(x)}{dx} \Delta x$.)

d. (4 points, a little harder) Suppose that the bank can trade in a ten-year zero-coupon bond $A$ with a face value of $100. It would like to take a position in bond $A$ that would offset (approximately) the interest rate risk exposure of XYZ. How many units of the bond should the bank buy or sell?

Solution

a. By the growing perpetuity formula,

$$P_{XYZ} = \frac{100}{0.06 - 0.02} = 2,500.$$

b. The general formula for the price is given by the growing perpetuity formula,

$$P_{XYZ} = \frac{100}{r - 0.02}.$$

c. The approximate change in price is given by

$$\Delta P_{XYZ} \approx \frac{dP_{XYZ}}{dr} \Delta r = -\frac{100}{(r - .02)^2} \Delta r.$$

d. For the ten year zero-coupon bond, the approximate change in value would be given by the duration formula,

$$dP_A \approx -\frac{1000}{{(1 + r)^4}}.$$
So, if \( x \) denotes the number of bond units we would like to buy, then

\[
xdP_A \approx \Delta P_{XYZ}
\]

or

\[
x \frac{1000}{(1 + r)^{11}} = \frac{100}{(r - .02)^2}
\]

and

\[
x = \frac{1}{10} \frac{(1 + r)^{11}}{(r - .02)^2}
\]

When \( r = 6\% \), we get \( x = 118.64 \).