Spinning of a molten threadline
Steady-state isothermal viscous flows

Jet equations and shape
The authors

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Motivation of the problem

- The main application is the understanding of fiber-drawing process of polymer melts, e.g. Dacron (poly(ethylene therephtalate)), polypropylene, Nylon (polyamide). VERY relevant industrially.

- Valuable information that we want with respect to the boundary conditions: radius, extension rate/jet shape (which has a strong influence on fiber properties)

- Also relevant is the stability of the jet (Pearson & Matovich 1969, *Spinning a Molten Threadline, Stability*), the stable operating space, and what parameters affect spinnability (=stability far from orifice).

- It can be extended to a lot of problems: non-isothermal, planar extrusion, steady jet on a planar surface...
Definition of the problem

We consider only that part

Variables

x

a(x)

x + dx

We consider only that part
Definition of the problem

We consider only that part

Variables

\[ \sin(\theta) \approx -a' \]
\[ \cos(\theta) \approx 1 + o(a') \]

Therefore

\[ \mathbf{n} = (r-a'x)/(1+a'^2)^{1/2} \quad (7) \]

(beware typo in paper)
Flow equations

- **Continuity**

  Div(\(\mathbf{u}\)) = 0 gives, in cylindrical coordinates:

  \[
  \frac{\partial v}{\partial x} + \frac{1}{r} \frac{\partial (ru)}{\partial r} = 0
  \]

  \[ (1) \]

- **Conservation of momentum**

  \[ \rho \left( \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} \right) = \rho f_i + \frac{\partial \tau_{ij}}{\partial x_j} \]

  in steady state and in cylindrical, gives:

  - **r-Momentum**

    \[
    \rho \left( \frac{u}{\partial r} + v \frac{\partial u}{\partial x} \right) = \frac{1}{r} \frac{\partial}{\partial r} (r \tau_{rr}) - \frac{1}{r} \tau_{\theta \theta} + \frac{\partial \tau_{rx}}{\partial x}
    \]

    \[ (2) \]

  - **x-Momentum**

    \[
    \rho \left( \frac{u}{\partial r} + v \frac{\partial v}{\partial x} \right) = \rho g + \frac{1}{r} \frac{\partial (r \tau_{rr})}{\partial r} + \frac{\partial \tau_{xx}}{\partial x}
    \]

    \[ (3) \]

  Here \(\tau\) is the total stress tensor, (usually written \(\sigma\), with \(\tau\) being the deviatoric stress)
Boundary conditions

• Radial BC (at $r=a(x)$):
  – Kinematic: the surface is a streamline, thus
    \[ \nu a' = u \]  
    \( (4) \)
  – Stress: free surface, no shear stress
    The Laplace pressure difference is $\sigma C$, where $C$ is the sum of the 2 curvatures: $1/a$ and $-a''/\sqrt{(1+a'^2)}^{3/2}$

\[
\sigma \left( \frac{1}{R} - \frac{1}{a} \right) n_r = \tau_{rr} n_r + \tau_{rx} n_x \\
\sigma \left( \frac{1}{R} - \frac{1}{a} \right) n_x = \tau_{rx} n_r + \tau_{xx} n_x
\]  
\( (5) \) (6)
Boundary conditions

- **Upstream and/or downstream BCs:**
  - imposed initial flow rate
  - plus one of the following:
    - Imposed final speed
    - Imposed final force

\[
\begin{align*}
\alpha &= a_0 \\
\left. v = v_0, \text{const.} \right|_{x = 0} \\
\end{align*}
\]

\[
\begin{align*}
v &= v_1, \text{const.}, \ \text{at} \ x = l \\
\left. 2\pi \int_0^{a_0} r x r dr = F_{x0} \right|_{x = 0} \\
\left. 2\pi \int_0^{a} r x r dr = F_{x1}, \ right|_{x = l} \\
\end{align*}
\]
Approximation scheme

- Development in power of $\alpha'$, which is $<<1$

\[
v = v^{(0)}(x) + v^{(1)}(r, x) + v^{(2)}(r, x) + \ldots
\]
\[
u = u^{(1)}(r, x) + u^{(2)}(r, x) + \ldots
\]
\[
p = p^{(0)}(x) + p^{(1)}(r, x) + p^{(2)}(r, x) + \ldots
\]
\[
\tau_{xx} = \tau_{xx}^{(0)}(x) + \tau_{xx}^{(1)}(r, x) + \tau_{xx}^{(2)}(r, x) + \ldots \quad (12)
\]
\[
\tau_{rr} = \tau_{rr}^{(0)}(x) + \tau_{rr}^{(1)}(r, x) + \tau_{rr}^{(2)}(r, x) + \ldots
\]
\[
\tau_{\theta\theta} = \tau_{\theta\theta}^{(0)}(x) + \tau_{\theta\theta}^{(1)}(r, x) + \tau_{\theta\theta}^{(2)}(r, x) + \ldots
\]
\[
\tau_{zx} = \tau_{zx}^{(1)}(r, x) + \tau_{zx}^{(2)}(r, x) + \ldots
\]
\[
\alpha = \alpha^{(0)}(x) + \alpha^{(1)}(x) + \alpha^{(2)}(x) + \ldots \quad (13)
\]

- Equations (22) through (30) are a proof of self-consistency, and a guide towards computing higher-order terms.
Approximation scheme (cont’d)

• Thin jet approximation: 0th-order term are independent of $r$

• 1st-order momentum equation

$$\rho v^{(0)} v^{(0)'} = \rho g + \tau_{xx}^{(0)'} + \frac{2 a^{(0)'}}{a^{(0)}} \left[ \tau_{xx}^{(0)} + \frac{\sigma}{a^{(0)}} \right]$$  \hspace{1cm} (20)

- The trick to easily derive (20) from (3) is to use the integral form, and retain only 1-order terms (top of page 515). That way, $a'$ shows up only in the change of area, and 1st and higher order terms of the expansion cancel out.

- $a$ and $a'$ are converted into $v$ and $v'$ using the conservation of flow rate (11)
Approximation scheme (cont’d)

Scaling of the different terms with a parameter $\epsilon$

\[ a_\epsilon(x) = \epsilon^\alpha a(x\epsilon^{-\gamma}) \]

\[ v_\epsilon(x) = \epsilon^\beta v(x\epsilon^{-\gamma}) \]

\[ v_\epsilon'(x) = \epsilon^{\beta-\gamma} v'(x\epsilon^{-\gamma}) \]

\[ a_\epsilon'(x) = \epsilon^{\alpha-\gamma} a'(x\epsilon^{-\gamma}) \]

(31)

(32)

Analogous solution:
terms must keep the same scaling even for $a' \rightarrow 0$

From this, they deduce the scaling of the parameters (33):

\[ \alpha > \gamma, \epsilon \rightarrow 0 \]

Relationship between $\alpha$ and $\beta$ given by $a'$ scaling and $\beta = \gamma$
Solutions

One need to provide a constitutive equation, then plug it into (20)

- **Newtonian case**
  - Constitutive equation
    \[ \mathbf{\varepsilon} = -\rho \mathbf{I} + \eta_0 \mathbf{e} \]  \hspace{1cm} (21)
  - Momentum equation
    \[ \rho \mathbf{\nu} \mathbf{v}' = \rho \mathbf{g} - 3\eta_0 \frac{(\mathbf{v}')^2}{\mathbf{v}} + 3\eta_0 \mathbf{v}'' - \sigma \pi^{1/2} \frac{v'}{2Q^{1/2}v^{1/2}} \]  \hspace{1cm} (34)

- The relative importance of the different terms is given by
  - Viscosity: 1
  - Inertia: Reynolds number \( \text{Re} \)
  - Gravity: Froude number \( \text{Fr} \), or gravity number \( B = \text{Re}/\text{Fr} \)
  - Surface tension: Weber number \( \text{We} \), or capillary number \( 1/\text{Ca} = \text{Re}/\text{We} \)
Solutions (cont’d)

- Newtonian case
  - Viscous-only solution (Re, Re/Fr, Re/We << 1)
    \[
    v(x) = v_0 \exp \left( \frac{x}{L_c} \right) \tag{37}
    \]
    \[
    L_c = \begin{cases} 
    \frac{l}{\ln \left( \frac{v_1}{v_0} \right)} \\
    \text{or } \frac{\eta_0 Q}{F_{t_0}} \\
    \text{or } \frac{\eta_0 Q}{F_{t_1}}
    \end{cases} \tag{38i, 38ii, 38iii}
    \]
  - Visco-inertial solution (Re ≈ 1, Re/Fr, Re/We << 1)
    \[
    v(x) = c_1 \left[ c_2 \exp \left( -c_1 x \right) - \frac{1}{3} \left( \rho / \eta_0 \right) \right]^{-1} \tag{39}
    \]

Depending on the BC

Sketch of the solutions for a_0 = 1mm, and arbitrary constants
Solutions (cont’d)

Newtonian case (cont’d)

- Visco-gravitational solution ($Re/\text{Fr} \approx 1$,
  \[ v(x) = \left(\frac{2\rho g}{3\eta_0 c_1}\right) \sinh^2 \left\{\frac{1}{2}c_1^{1/2}(x + c_2)\right\} \]  

  Comes from Trouton, (1906). Determining the constants $c_1$ and $c_2$ is easier said than done...

  Ribe (2004) gives another solution, for the BC (i), which has a small range of application:
  \[ v = v_1 \cos^2\left(\sqrt{\frac{\rho g Q}{\nu_1 (x + x_1)}}\right) \]

- Viscosity and surface tension ($Re/\text{We} \approx 1$, $Re, Re/\text{Fr} << 1$)

  \[ v(x) = \left[\frac{c_2}{c_1} \exp\left(\frac{1}{2}c_1 x\right) + \frac{\sigma \pi^{1/2}}{3\pi_0 Q^{1/2}c_1}\right]^2 \]  

- Inviscid solutions ($Re, Re/\text{Fr}, Re/\text{We} >> 1$) are not of concern here. They can be found for example in *The Mechanics of Liquid Jets*, by J.N. Anno.
Solutions (cont’d)

• Non-Newtonian case: a lot of models are available
  – A simple one is the inelastic fluid model: the Newtonian viscosity is replaced by a Trouton viscosity

\[ \tau_{xx} = -\left(\sigma/\alpha\right) + \eta_T(v')v' \]  \hspace{1cm} (44)

This gives a momentum equation of the form (45):

\[ \rho uv' = \rho g - 3\eta_T \frac{(u')^2}{v} + 3(\eta_T + u'\eta_T')v'' - \sigma \pi^{1/2} \frac{v'}{2Q^{1/2}v^{1/2}} \]

Here again, different models for the viscosity. The simplest is the power-law model:

\[ \eta_T = \eta_D (v')^{q-1} \]  \hspace{1cm} (46)

The solution is easy for viscous-only case:

\[ \bar{v}(x) = \left\{ v_0^m + (v_1^m - v_0^m) \left( \frac{x}{L} \right) \right\}^{1/m} \]

\[ m = \frac{(q - 1)}{q} \]  \hspace{1cm} (47)

As expected, shear-thinning hinders spinnability.
Non-Newtonian case: a lot of models are available

- A second step towards difficulty is the second-order fluids model:
  \[ \tau = -p I + \eta_0 d - \eta_1 d + \eta_2 d^2, \quad (52) \]

This leads to a third-order differential equation for the conservation of momentum:

\[
\rho v' = \rho g + 3\eta_0 \left[ v'' - \frac{(v')^2}{v} \right] + 3\eta_0 \varepsilon \left[ 2v'v'' - \frac{(v')^3}{v} \right] - vv''' + \frac{\sigma a'}{a^2} \quad (53)
\]

- To solve it, they use an expansion in powers of a Deborah number \( \Delta \)

In dimensionless form,

\[
\psi = \psi_0 + \Delta \psi_1 + \Delta^2 \psi_2 + \ldots + \Delta^n \psi_n + 0(\Delta^n) \quad (57)
\]

Then, every order gets its own equation (and needs its own 2 BCs…)

0-order:

\[ \psi_0 \psi_0'' - (\psi_0')^2 = 0 \quad (58.0) \]

1st-order:

\[ \psi_0 \psi_1'' - 2\psi_0 \psi_1' + \psi_0'' \psi_1 = \xi \left\{ (\psi_0')^3 - 2\psi_0 \psi_0' \psi_0'' \right\} + \psi_0^2 \psi_3'' \quad (58.1) \]

n-th-order:

\[ \psi_0 \psi_n'' - 2\psi_0 \psi_n' + \psi_0'' \psi_n = f_n(\psi_{n-1}, \ldots, \psi_0) \quad (58.n) \]
Solutions (cont’d)

• Non-Newtonian case

They give the solutions for the first two orders

0-order: \[ \psi_C \psi_0'' - (\psi_0')^2 = 0 \]  \hspace{1cm} (58.0)

gives

\[ \psi_0 = C_1 e^{C_2 x} \]  \hspace{1cm} (59)

1st-order: \[ \psi_0 \psi_1'' - 2\psi_0 \psi_1' + \psi_0'' \psi_1 = \xi \{ (\psi_0')^3 - 2\psi_0 \psi_0' \psi_0'' \} + \psi_0^2 \psi'' \]  \hspace{1cm} (58.1)

gives

\[ \psi_1 = \begin{cases} 
(1 - \xi) \left\{ e^{2x} - e^x - \frac{(v_1 - v_0)Xe^x}{v_0 \ln(v_1/v_0)} \right\} & \text{(64i)} \\
(1 - \xi) \left\{ e^{2x} - (1 + X)e^x \right\} & \text{(64ii)} \\
(1 - \xi) \left\{ e^{2x} - e^x - e^2Xe^x \right\} & \text{(64iii)} 
\end{cases} \]

Depending on the BC

(65) through (71) discuss the validity of the solution, depending on the BCs (the perturbation method loses ground for \( \Delta \) too large) and give another derivation route.
Extension to Nonisothermal flows

One needs:

- An equation of state (which can be T-dependent)
- To include temperature convection in flow equation

\begin{equation}
\rho C_p \left( u \frac{\partial T}{\partial r} + v \frac{\partial T}{\partial x} \right) = k \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial x^2} \right] + \\
\left[ \tau_{rr} \frac{\partial u}{\partial r} + \tau_{\theta \theta} \frac{u}{r} + \tau_{xx} \frac{\partial v}{\partial x} + \tau_{r\theta} \left( \frac{\partial v}{\partial r} + \frac{\partial u}{\partial x} \right) \right]
\end{equation}  \quad (72)

- One radial and two axial boundary conditions for temperature. The most obvious is \( T = T_1 \) for the melt reservoir, \( T = T_0 \) for the ambient air, and a flux at the interface proportional to \( T - T_0 \) ((73) to (75)).
Extension to Fiber drawing and Film casting stability

• Jet stability:
  Pearson & Matovich 1969, *Spinning a Molten Threadline, Stability*. They take in account different causes of instability: radius or speed varying at the origin, speed or tension varying at the wind-up (but they don’t take in account extension thickening, which should play a role in stabilizing…).

• Film casting:
  Yeow (J. Fluid Mech., 1974). Their problem is no longer axisymmetric.
Extension to jet on a plate

- **Steady jet:**
  Cruikshank and Munson (1982). “v=0 at the plate” boundary condition.

- **Coiling jet:**
  The speed at the plate is **non-zero, non-imposed: we lose a boundary condition.**
Three different problems

- **Matovitch & Pearson**: Drawn fiber, ie final speed or force imposed.
- **Cruikshank & Munson**: Steady jet on a plate, ie speed = 0 at the plate.
- **Our problem**: Non-steady jet on a plate, ie, non-imposed, non-zero speed at the plate.