a) \[ \frac{1}{2} L \sigma_x = \frac{1}{2} \sigma_x \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
\[ \frac{1}{2} L \sigma_z = \frac{1}{2} \sigma_z \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

\[ S_x + S_z = \frac{1}{2} \sigma_x + \frac{1}{2} \sigma_z = \frac{1}{2} \left( \sigma_x + \sigma_z \right) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \]

Eigenvalues are \( \frac{1}{2} \) x eigenvalues of \( \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \):

\[ 0 = \det \left( \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} - \lambda I \right) = (1 - \lambda)(-1 - \lambda) - 1 \]
\[ = -(1 - \lambda^2) - 1 = \lambda^2 - 2 \Rightarrow \lambda = \pm \sqrt{2} \]

Thus, possible results are \( \pm \frac{\sqrt{2}}{2} \)

b) The eigenvectors \( \left( \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \right) \) obey

\[ (1 - \lambda) \alpha + \beta = 0 \]
\[ (1 + \sqrt{2}) \alpha + \beta = 0 \]

If the eigenvalue is \( \sqrt{2} \),

So, unnormalized, the state after measurement \( \sqrt{2} \) is

\[ \left( \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} \right) \]

So that

\[ P_{S_x = \frac{1}{2}} = \frac{1}{1 + (1 - \sqrt{2})^2} = \frac{1}{4 - 2\sqrt{2}} \]
\[ P_{S_z = -\frac{1}{2}} = \frac{(1 - \sqrt{2})^2}{1 + (1 - \sqrt{2})^2} = \frac{3 - 2\sqrt{2}}{4 - 2\sqrt{2}} \]

After measuring \( S_x + S_z = m \frac{1}{2} \), \( m = \pm 1 \)
Section I: Invoking Quantum Mechanics

I-2 Electron Pressure

An electron is inside a sphere of radius $R$. What pressure $P$ does it exert on the wall, if it is:

i) in the lowest $S$-state?

ii) in the lowest $P$-state? [Hint: $\psi_1 \sim \frac{d\psi}{d(kR)}$, solve approximately]

Shn I-2

Shn. Solve Schröd to get $E(R)$

- expand sphere by $dR$

Work done is: $dW = P \cdot dV = 4\pi r^2 P dR = -dE(R) = -\frac{dE(R)}{dR} dR$

Hence:

$$P = -\frac{dE}{dR} \frac{1}{4\pi^2}$$

i) lowest $S$-state: $\psi_1 \sim \frac{m^2}{2}\frac{1}{R} \quad \text{nn:} \quad \psi_1(R) = 0 \implies kR = \pi$

Thus:

$$E = \frac{\hbar^2 \pi^2}{2m}, \quad \frac{\pi^2 \hbar^2}{2m R^2}$$

$$\implies \boxed{P = \frac{\pi^2 \hbar^2}{4 m R^2}}$$

ii) lowest $P$-state: $\psi_1 \sim \frac{\cos(kR)}{kR} - \frac{m^2}{(kR)^2} = \frac{d\psi}{d(kR)}$

$\psi_1(R) = 0 \implies kR \cot(kR) = \theta \quad \text{solve by iteration} \quad kR \approx 4,5$

$$\implies \boxed{P \approx (4,5) \frac{\hbar^2}{4 m R^2}}$$
I - 3 States and Observables of a System

Observable $A$ has eigenstates $|\pm\rangle$, where $A|\pm\rangle = \pm |\pm\rangle$. The Hamiltonian for this system is defined by $\langle +|H|-(\rangle = \hbar \omega$, and $\langle +|H|+(\rangle = \langle -(|H|-(\rangle = 0$.

a. Find the normalized eigenstates of $H$ in terms of the states $|\pm\rangle$, and the corresponding eigenvalues.

b. What are the matrix representations of $A$ and $H$ in the $H$ eigenstates basis?

c. Given that $|\Psi(t=0) = +\rangle$, what is $|\Psi(t)\rangle$ for any $t$?

d. Averaged over many identical experiments, what would one measure for the observables $A$ and $H$ when the system is in the state $|\Psi(t)\rangle$?

e. What is the probability of obtaining the result $-1$ in a measurement of $A$ at time $t$?

f. At time $t=10$, for example, the Hamiltonian suddenly changes to $H = \hbar \omega A$. What will $|\Psi(t)\rangle$ be at $t = 20\omega$?

---

Sln 1-3

a) In the $|\pm\rangle$ basis $|\pm\rangle$: $H = \begin{pmatrix} 0 & \pm \hbar \omega \\ \pm \hbar \omega & 0 \end{pmatrix}$

Eigenstates from $\det (H - \lambda I) = 0 \Rightarrow \lambda^2 + \hbar^2 \omega^2 = 0 \Rightarrow \lambda = \pm \hbar \omega$

$\lambda_+ = + \hbar \omega \Rightarrow (\pm \hbar \omega - \hbar \omega, \pm \hbar \omega)(\pm \omega) = 0 \Rightarrow \alpha \pm i \beta = 0$

$e_+ = \frac{1}{\sqrt{2}} (1,0)$ (Normalized) $\frac{\hbar \omega}{\sqrt{2}}$

$\lambda_- = - \hbar \omega \Rightarrow (\pm \hbar \omega + \hbar \omega, \pm \hbar \omega)(\pm \omega) = 0 \Rightarrow \alpha \pm i \beta = 0$

$e_- = \frac{1}{\sqrt{2}} (-1,0)$ (Normalized) $\frac{1 - \hbar \omega}{\sqrt{2}}$

---
6) \( H_{e^+} = \frac{\hbar w}{e^+} \Rightarrow H = \begin{bmatrix} \hbar w & 0 \\ 0 & -\hbar w \end{bmatrix} \)

\( \langle \psi_n | H | \psi_m \rangle = \hbar w \delta_{nm} n \), \( n,m \geq 2 \)

\( A e^+ = \frac{1}{\sqrt{2}} (\frac{1}{i}) e^- \Rightarrow A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) in \( e^+ \) basis.

\( A e^- = \frac{1}{\sqrt{2}} (\frac{1}{i}) e^+ \)

\( \langle \psi_n | A | \psi_m \rangle = (1 - \delta_{nm}) n,m \geq 1 \)

4) \( \psi(t=0) = 1+ = \frac{1}{\sqrt{2}} [\hbar \omega > + 1 - \hbar \omega >] \)

\( \psi(t) = \frac{1}{\sqrt{2}} (\hbar \omega e^{-i\omega t} + 1 - \hbar \omega) e^{i\omega t} = \frac{e^{-i\omega t} + i e^{-i\omega t}}{2} \cos \omega t + \sin \omega t \)

3) \( \langle E \rangle = \langle \psi(t) | \frac{\hbar^2}{2} e^{-i\omega t} | \hbar^2 e^{i\omega t} \rangle = \frac{1}{2} e^{-i\omega t} - \frac{1}{2} e^{i\omega t} = 0 \)

\( \langle A \rangle = (1) \cos^2 \omega t + (-1) \sin^2 \omega t = \cos^2 \omega t - \sin^2 \omega t = \cos 2\omega t \)

2) \( P_{A=1} = |\sin \omega t|^2 = \sin^2 \omega t \)

1) \( \psi(t=0) = \cos (\omega t) |+\rangle + \sin (\omega t) |-\rangle \)

\( \psi(t > 0) = \cos (\omega t) e^{-i\omega t} |+\rangle + \sin (\omega t) e^{i\omega t} |-\rangle \)

\( \psi(t > 0) = \cos (\omega t) e^{-i\omega t} |+\rangle + \sin (\omega t) e^{i\omega t} |-\rangle \)

\( \psi(t = \infty) = \cos (\omega t) e^{-i\omega t} |+\rangle + \sin (\omega t) e^{i\omega t} |-\rangle \)
I - 4 Sudden Change in Potential

For t<0, an electron is in the ground state of the one dimensional potential \( V(x) = A\delta(x) \). At t=0, the potential suddenly changes to \( V'(x) = A'\delta(x) \). What is the probability that the electron will be in the ground state of the potential \( V' \) for times \( t>0 \)?

For:

\[ E \quad \downarrow \quad x \rightarrow \]

\[ V'' - A\delta(x) \]

We have:

\[ \frac{-h^2}{2m} \psi'' - A\delta(x) \psi = E \psi \]

\[ \psi'' + \frac{2mA}{h^2} \delta(x) \psi = \frac{-2mE}{h^2} \psi = x^2 \psi \]

For \( x \neq 0 \), this is:

\[ \psi'' = x^2 \psi \]

\[ \implies \psi = a e^{x^2} + b e^{-x^2} \]

Boundary conditions: To be normalized, \( \psi \) exponentially growing.

\[ \psi(x) = \begin{cases} 
\alpha e^{x^2} & x > 0 \\
\beta e^{-x^2} & x < 0 
\end{cases} \]

\( \psi \) is continuous at \( x=0 \) \( \alpha = \beta \) \( \psi(x) = \alpha e^{x^2} \]

\[ \int \psi'' = \frac{2mA}{h^2} \delta(x) \psi dx = \int x^2 \psi dx \]

\[ (\psi'/4 - \psi/4) + \frac{2mA}{h^2} \psi = 0 \]

\[ (-k\psi' - (k^2)x + \frac{2mA}{h^2} \psi) = 0 \]

\[ \implies \psi = \frac{m A}{k} \]

\[ \frac{m \psi}{k} \]
Normalizing:

\[ 1 = \int_{-\infty}^{\infty} |\psi|^2 \, dx = \int_{-\infty}^{\infty} x^2 e^{-2|x|} \, dx \]
\[ = 2\int_{0}^{\infty} x^2 e^{-2x} \, dx \]
\[ = \frac{2\sqrt{\pi}}{2} \]

\[ \Rightarrow \alpha = \sqrt{\frac{2}{\alpha}} \]

\[ \Rightarrow \psi(x) = \sqrt{\frac{m}{2\pi \hbar^2}} e^{-\frac{mA}{4x} |x|} \]

The probability is the square of the amplitude \( \psi \):

\[ P = 1 |\psi|^2 \]
\[ = \int dx \psi \psi^*(x) \psi(x) \]
\[ = \int dx \sqrt{\frac{m}{2\pi \hbar^2}} e^{-\frac{mA}{4x} |x|} \sqrt{\frac{m}{2\pi \hbar^2}} e^{-\frac{mA}{4x} |x|} \]
\[ = \frac{m}{4\pi} \int dx \frac{1}{\sqrt{A^*A}} e^{-\frac{mA}{2\hbar^2} (A^2 + A^{*2})} \]
\[ = \frac{m}{4\pi} \sqrt{A^*A} \int dx \frac{1}{\sqrt{A^*A}} e^{-\frac{mA}{2\hbar^2} (A^2 + A^{*2})} \]
\[ = \frac{m}{4\pi} \sqrt{A A^*} \cdot \frac{1}{\sqrt{A^* A}} = 2 \sqrt{\frac{AA^*}{(A+A^*)^2}} \]

\[ \Rightarrow P = 4 \frac{AA^*}{(A+A^*)^2} \]
Shr II - 1 Field of a current loop.

a) Field of the loop \( \mathbf{B} \)

\[
B = \frac{\mu_0 I d x}{4\pi} \cdot \frac{1}{r^2} \cdot \frac{2\pi \hat{b}}{(a^2 + z^2)} \cdot \frac{b}{\sqrt{a^2 + z^2}} = \frac{\mu_0 I b^2 L}{2\sqrt{a^2 + z^2}}.
\]

b) Small \( \Delta y \) off axis

Use: \( \text{div} \mathbf{B} = 0 \) \( \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0 \)

Note: From b) cylindrical symmetry

\( \Rightarrow \) use \( \nabla \cdot \mathbf{B} = 0 \)

\( r = \sqrt{x^2 + y^2} \)

small \( \Delta y \):

\[
\frac{\partial B_y}{\partial y} = \frac{\partial B_z}{\partial z} = \frac{\partial B_x}{\partial x} = \frac{\partial B_y}{\partial y} = \frac{\partial B_z}{\partial z}.
\]

\[
B_y = \frac{\partial B_y}{\partial y} \Delta y = -\frac{\partial B_z}{\partial z} \Delta y = -\frac{\Delta y}{2\pi} \frac{I b^2}{\sqrt{a^2 + z^2}} \Delta y.
\]

\( \text{div} \mathbf{B} = 0 \)
II - 2 Rotating Ring

A metal ring rotates in a weak homogeneous magnetic field $B$ around an axis perpendicular to $B$, see sketch. Due to Joule heat dissipation it slows down to $\omega_0 e^{-t}$ in a time $\tau$ from its initial value $\omega_0$.

Calculate $\tau$ for a ring made from a round wire of a metal with resistivity $\eta (\Omega \cdot \text{m})$ and density $\rho \text{ (g/cm}^3\text{)}. Neglect self induction and assume that only a small amount of energy is lost in one revolution.

\[ \text{Sln II-2} \]

Flux $\Phi = \Phi_0$: \[ \Phi = B \cdot \pi a^2 = B \pi a^2 \cos \omega_0 t \]

Faraday: \[ \varepsilon = -\frac{d\Phi}{dt} = -B \pi a^2 \sin \omega_0 t = IR \]

with \[ R = \frac{\eta}{\pi b^2} \text{ (} b \text{ is radius of wire)} \]

kin. E \[ E = \frac{1}{2} \Theta \omega^2 \text{ with moment of inertia } \Theta \]

\[ \Theta = \int r^2 dm = \int (\rho \frac{a}{2} \sin \theta)^2 dl \cdot \pi b^2 \rho \]

\[ \Theta = \pi a^2 b^2 \int_0^{\pi} \sin^2 \theta d\theta = \pi^2 a^2 b^2 \rho \]

\[ \text{nok: } \Theta = \frac{ma^2}{2} \]

Energy conservation: \[ \frac{d}{dt} \cdot \frac{ma^2}{2} \omega^2 = -\frac{\Theta^2 \omega^2}{2R} = \frac{d\varepsilon}{dt} \]

Diff eq: \[ \Theta \omega \ddot{\omega} = -\frac{\Theta^2 \omega^2}{2R} \]

\[ \text{Sln: } \]
\[ \omega = \omega_0 \exp \left(-\frac{a^2 \pi^2 \omega^2}{2R}\right) \]

\[ \tau = \frac{2R \Theta}{\eta a^2} = \frac{2\pi \eta b^2}{\pi^2 a^2 b^2} = \frac{4\eta \theta}{a^2} \]

Capacitance of plate.

\[ C = \frac{Q}{V} \]

DC capacitance is \( C_0 \)

\[ V = V_0 e^{i\omega t} \]

(a) Electric field?

By definition,

\[ E = \frac{V}{\mu t} = \frac{V_0}{d} \]

(b) Magnetic field between plates?

\[ \oint B \cdot dl = \mu_0 \int \left( \frac{j}{\varepsilon_t} + \varepsilon_0 \frac{\partial E}{\partial t} \right) \cdot dl \]

\[ B \cdot 2\pi r = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t} \cdot 2\pi r \]

\[ B = \frac{\mu_0 \varepsilon_0}{2} \frac{\partial E}{\partial t} \]

\[ B = \frac{\mu_0 \varepsilon_0}{2} \frac{V_0}{d} (i\omega) e^{i\omega t} \]

(c) First order correction to electric field?

Now \( \nabla \times \vec{E} = -\frac{j B}{c^2} \)

\[ \vec{E} = (0, 0, E_z) \]

\[ \frac{\partial E_z}{\partial z} - \frac{\partial E_y}{\partial y} = \frac{-dB_z}{dt} \]

\( \text{appropriate on y axis} \)

where \( B = (B_x, 0, 0) \)

\( \nabla \times \vec{B} = 0 \)

\[ \frac{\partial E_z}{\partial y} = \frac{1}{c^2} \frac{-dB_z}{dt} \]

\( \text{where} \ B_x = \frac{1}{c^2} \frac{\varepsilon_0 V_0}{d} (i\omega) e^{i\omega t} \)

\[ \frac{\partial E_z}{\partial y} = \frac{\mu_0 \varepsilon_0 V_0}{2} \frac{w^2 y^2}{d^2} e^{i\omega t} \]
(a) Charge density \text{(net)} \text{ on plates}:

\[ E = E_0 + E_{\text{vac}} = \frac{V_0 e^{i \omega t}}{d} \left[ \frac{1 + \frac{i \mu_0 \omega}{4 \epsilon_0}}{1 + \frac{i \mu_0 \omega}{4 \epsilon_0}} \right] \]

From Gauss's law:

\[ \int E \cdot ds = \frac{Q_{\text{enc}}}{\epsilon_0} \]

\[ E \cdot ds = \frac{\sigma ds}{\epsilon_0} \]

\[ E = \frac{\sigma}{\epsilon_0} \]

\[ \therefore \sigma = E \epsilon_0 \]

\[ \sigma = \frac{V_0}{d} \epsilon_0 a e^{i \omega t} \left[ 1 + \frac{i \mu_0 \omega}{4 \epsilon_0} \right] \]

(b) Effective capacitance:

\[ C = \frac{Q}{V} \text{ where } Q = \int \sigma(a) ds = \int (\sigma(a), 2\pi a) da \]

\[ Q = \frac{V_0}{d} \epsilon_0 e^{i \omega t} \frac{\pi a}{1} + \frac{V_0}{d} \epsilon_0 e^{i \omega t} \frac{\mu_0 \epsilon_0 \omega}{4} 2\pi \int R^2 dr \]

\[ C = \frac{Q}{V_0 e^{i \omega t}} = \frac{\epsilon_0 \pi a^2}{d} + \frac{\epsilon_0 (\mu_0 \epsilon_0 \omega^2) \pi a^2}{2} \]

\[ C = \frac{\epsilon_0 \pi a^2}{d} \left[ 1 + \frac{\mu_0 \epsilon_0 \omega^2 a^2}{8} \right] \]
II - y

(a) on y axis, only +2Q, zero is at y = a, by inspection.

\[ \frac{x^2}{y^2} - \frac{(x-a)^2}{(y-a)^2} = 0 \implies y^2 - 4ay + 2a^2 \]

\[ y = (2 \pm \sqrt{2}) a \]

\[ y \approx 0.6 \alpha, 3.4 \alpha \] excluded

(b) one half of field lines from 2Q must terminate on -Q. These will be the field lines starting in directions closest to that of the direction to -Q, i.e. those with \( \theta < 90^\circ \). For \( \theta > 90^\circ \), lines terminate at \( \infty \).

\[ (x) \]
Section III: Involving Statistical Mechanics and Thermodynamics

III - 1 Heat Capacity

Consider a solid piece of material containing $N$ nuclei of spin 1, which do not interact. Each nucleus can be in $m = 0$ or ±1 state. Due to the internal electric field in the solid the $m = ±1$ states have the same energy $\varepsilon > 0$, while $m = 0$ has energy 0.

Deduce the entropy of the $N$ nuclei as a function of temperature, and give the heat capacity for $\varepsilon/kT << 1$.

Solution III - 1

\[ Z = \sum e^{-\frac{\varepsilon}{kT}} = (1 + 2 e^{-\frac{\varepsilon}{kT}}) \]

\[ F = -kT \ln Z = -NkT \ln (1 + 2 e^{-\frac{\varepsilon}{kT}}) \]

\[ S = -\left( \frac{\partial F}{\partial T} \right)_{V,N} = kN \ln (1 + 2 e^{-\frac{\varepsilon}{kT}}) + \frac{2N\varepsilon}{T} \frac{e^{-\frac{\varepsilon}{kT}}}{(1 + 2 e^{-\frac{\varepsilon}{kT}})} \]

\[ E = U = TS \quad U = \frac{2N\varepsilon e^{-\frac{\varepsilon}{kT}}}{(1 + 2 e^{-\frac{\varepsilon}{kT}})} \approx \frac{2N\varepsilon}{3} \left( 1 - \frac{\varepsilon}{3kT} \right) \quad \text{for } \frac{\varepsilon}{kT} << 1 \]

\[ C = \frac{\partial U}{\partial T} = \frac{2N\varepsilon^2}{3} \frac{1}{kT^2} \approx \frac{2}{3} kN \left( \frac{\varepsilon}{kT} \right)^2 \]
\[ L = \frac{\pi}{2} \]

\[ \omega = \frac{c}{k} \]

\[ k = \frac{\omega}{c} \]

\[ n = \frac{\omega}{\omega_0} \]  

\[ n = \frac{L}{\pi} \]

\[ \omega_0 = \frac{c}{\pi} \]

(a) \[ \frac{N}{2} \] - allowed values for \( \omega \), \( \omega < \frac{\pi}{2} \)

\[ f(k) = \left( \frac{k}{\omega} \right) \frac{1}{\pi \sqrt{1 - k^2}} \]

(b) \[ f(w) \, dw = 2 f(k) \, dk \]

\[ \therefore f(w) = 2 f(k) \cdot \frac{dk}{dw} = 2 \left( \frac{L}{2\pi} \right) \cdot \frac{1}{\sin x} = \frac{L}{\pi c} \]

(c) \[ \omega_0 = \frac{c}{k} \]

\[ \{ k \} = \frac{X}{X} \]

\[ \frac{N}{\omega_0} \]

\[ f(w) = \frac{1}{\pi} \frac{N}{\omega_0} = \frac{N}{\omega_0} \]
\[ U = \int_0^{\omega_d} \left( \omega \phi(\omega) + N \phi(\omega) \right) d\omega \]

\[ \bar{E} = \frac{\omega_d}{2} + \frac{\omega_d}{2} \]

\[ e^\omega = \frac{\omega_d}{\hbar} = \chi \]

\[ \frac{N}{\omega_d} \int_0^\chi \frac{e^x - 1}{x} dx + \int_0^\chi \frac{N}{\omega_d} \frac{e^x - 1}{x} dx = \frac{\pi^2}{6} \]

\[ = \frac{N!}{2} \frac{\omega^2}{2 \omega_d} - \frac{\min^2}{6} + \frac{\pi^2}{6} \frac{N!}{\omega_d} \frac{\omega^2}{\hbar} + \frac{\pi^2}{6} \frac{N!}{\omega_d} \frac{(k_B T)^2}{\hbar^2} \]

For \( T \to 0 \), the first term \((o.s.)\) becomes relevant.

\[ C_V = \frac{\partial U}{\partial T} = \frac{\pi^2}{6} \frac{N^2}{\hbar \omega_d} \cdot 2k_B^2 T \]

\[ = \frac{\pi^2}{3} \frac{N^2}{(\hbar \omega_d)} k_B^2 T \]
III - 3 Cooling by relaxing B field.

At low temperatures one can cool further by reducing the magnetic field penetrating a paramagnetic substance. Assume \( M = \alpha(T)B \), i.e. the magnetization is linear proportional to the B field applied. Find the temperature change \( \Delta T \) for a magnetic field decrease \( \Delta B \) in a thermally isolated sample of heat capacity \( c_B \) at constant \( B \).

**Sln III-3**

**Sln:** Suppose \( M = \alpha(T)B \),

we have \( dU = TdS + BdM \)

**Thermal isolated:** \( \Delta S = 0 \)

Then: \( \Delta T = \left. \frac{\partial T}{\partial H} \right|_S \Delta H \)

\[
\alpha(U - BM) = TdS - MdB
\]

\[
\frac{\partial^2(u - B\alpha)}{\partial S \partial B} = \left. \frac{\partial T}{\partial B} \right|_S = -\left. \frac{\partial M}{\partial S} \right|_B
\]

\[
\Delta T = -\left. \frac{\partial M}{\partial S} \right|_B \Delta B = -\left. \frac{\partial M}{\partial T} \right|_B \frac{\partial S}{\partial T} \Delta B
\]

\[
\Delta T = -\frac{\partial \alpha}{\partial T} B \Delta B
\]

\[
= -\frac{T}{c_B} \frac{\partial \alpha}{\partial T} B \Delta B
\]
III - LC Thermometer

By measuring the noise (rms) voltage across a capacitor in parallel with an inductor one can determine the temperature, T. Find the relation between T and \( V_{\text{rms}} = \langle \delta V^2 \rangle^{1/2} \).

Start with a Hamiltonian involving Q:

\[
H = \frac{1}{2} L \left( \frac{dQ}{dt} \right)^2 + \frac{Q^2}{2C} \quad \rightarrow \quad \text{ham. vars} \quad \omega = \frac{1}{\sqrt{LC}}
\]

\[
E_n = \frac{\hbar \omega}{2} (n + \frac{1}{2})
\]

- in circuit

\[
U = \langle E \rangle = \frac{\Sigma E_n}{N} \cdot \frac{E_n}{kT} = \frac{\hbar \omega}{2} + \frac{\hbar \omega}{e^{\frac{\hbar \omega}{kT}} - 1}
\]

\[
\omega_{\text{circuit}} = \langle CV^2 \rangle = \langle \frac{L}{2} \rangle
\]

\[
\langle V^2 \rangle = \frac{\hbar \omega}{2C} \left\{ \frac{e^{-\frac{E}{kT}}}{e^{-\frac{E}{kT}} - 1} \right\}
\]

\[
= \frac{\hbar \omega}{2C} \cosh \left( \frac{\hbar \omega}{kT} \right)
\]

- In the limit: \( kT \gg \hbar \omega \):

\[
\langle V^2 \rangle = \frac{kT}{2C}
\]

- Otherwise: \( kT \ll \hbar \omega \):

\[
\langle V^2 \rangle = \frac{\hbar \omega}{2C}
\]
Section IV: Classical Physics of Mechanics

IV - 1 Chimney Breaking Point

A thin uniform brick chimney falls, pivoted by its low end. Consider the flexion stress at a section through the chimney and calculate the likely point of rupture.

Consider lower portion:
\[ \frac{M}{3} \frac{d^2 \theta}{dx^2} = -Mg \frac{x}{L} \sin \theta \]

\[ \Rightarrow \frac{d^2 \theta}{dx^2} = \frac{-3Mg}{2L} \sin \theta \]

Rotation of upper portion:
about its center of mass:
\[ \frac{M \left( L - x \right)^2}{12L} \frac{d^2 \theta}{dx^2} = \frac{\left( L - x \right)^3}{L} - \tau \]

Solve 2 eq:
\[ \tau = \frac{Mg \sin \theta \left( L - x \right)}{2x (L + x)} \left( \frac{L^2}{3} - x^2 \right) \]

\[ \tau = \frac{Mx \left( L - x \right)^2 \sin \theta}{4 + L^2} \]

Since the chimney is thin width w:

Breaking at:
\[ \tau_{\text{max}} \frac{d^2 \theta}{dx^2} = \left( L - x \right)^2 = 2x \left( L - x \right) = 0 \]

\[ 1^2 - 2Lx + x^2 - 2Lx + 2x^2 = 0 \]

\[ L^2 - 4Lx + 3x^2 = 0 \]

\[ x^2 - \frac{4L}{3}x + \frac{L^2}{3} = 0 \]

\[ c_{\text{min}} = \left( \frac{2}{3} \pm \sqrt{\left( \frac{2}{3} \right)^2 - 1} \right) \]

\[ = \frac{1}{2} L \]
1. Jet Engines

a. The mass of area flowing into the intake per unit time is \( \dot{m} = \frac{dn}{dt} = \nu_1 \rho A \).

b. The force applied to the engine equal to the momentum transferred to the air per unit time (Newton's Second Law), \( F = \dot{m}(v_r - v_i) = \dot{m} \Delta v \). The power output of the engine is the energy output per unit time, \( P = \dot{E} = \eta \dot{m} \Delta v v_i \).

c. The kinetic energy increase per unit time of the air flowing in is \( \dot{E}_{kin} = \frac{1}{2} \dot{m}(v_r^2 - v_i^2) \).

d. The propulsive efficiency \( \eta_p \) is the ratio of the power output of the engine \( P \) to the energy increase per unit time of the gas \( \dot{E}_{kin} \),

\[
\eta_p = \frac{\dot{m} \nu \Delta v}{\frac{1}{2} \dot{m}(v_r^2 - v_i^2)} = \frac{2}{2 + \frac{\Delta v}{\nu_1} \frac{v_i}{v_r}}
\]

e. The mass flow is now \( \dot{m}' = \nu_2 \rho A' \). The force is the same as in part b, \( (v_r - v_i) = \alpha (v_r' - v_i) \); in order to achieve the same power output, more air is moved at a lower velocity. The propulsive efficiency is then

\[
\eta_p' = \frac{\dot{m} \nu \Delta v'}{\frac{1}{2} \dot{m}(v_r'^2 - v_i^2)} = \frac{2}{2 + \frac{\Delta v'}{\nu_2} \frac{v_i}{v_r}} = \frac{2}{2 + \frac{\Delta v}{\alpha \nu_1}}
\]
For a loop:

\[ I_{xx} = \int \rho \, d\phi \, \frac{m}{2\pi R} \left( R^2 - R^2 \cos^2 \phi \right) \]

\[ = \frac{mR^2}{2\pi} \int_0^{2\pi} \sin^2 \phi \, d\phi \]

A trig identity:

\[ \sin^2 \theta = -\frac{1}{4} \left( e^{2i\theta} - 2 + e^{-2i\theta} \right) = \frac{1}{2} - \frac{1}{2} \cos 2\theta \]

\[ I_{xx} = \frac{mR^2}{2\pi} \left( \frac{1}{2} \right) \int_0^{2\pi} (1 - \cos 2\theta) \, d\theta \]

\[ = \frac{mR^2}{4\pi} \left( 2\pi - \frac{1}{2} \int_0^{2\pi} \cos 2\theta \, d\theta \right) = \frac{mR^2}{2} \]

\[ I_{xx} = I_{yy} \]

\[ I_{zz} = \int_0^{2\pi} R \, d\phi \, \frac{m}{2\pi R} \rho^2 = mR^2 \]

\[ I_{xy} = \int_0^{2\pi} R \, d\phi \, \frac{m}{2\pi R} (-xy) \]

\[ = \frac{m}{2\pi} \left[ \int_0^{2\pi} \sin \theta \cos \theta \, d\theta + \frac{m}{2\pi} \int_0^{2\pi} \cos \theta \, d\theta \right] \]

\[ = \frac{m}{2\pi} \left[ \frac{1}{2} \sin^2 \theta \right]_0^{2\pi} = 0 \]

\[ I_{x2} = \frac{m}{2\pi} \int_0^{2\pi} \rho \, d\phi \, \frac{m}{2\pi R} \left( -\chi \rho \right) = I_{y2} = 0 \]
For loop 1:

\[ I_{xx_1} = I_{xy_1} = \frac{MR^2}{2} \]

\[ I_{zz_1} = MR^2 \]

\[ I_{yy_1} = I_{y_1y_1} = I_{y_2} = 0 \]

For loop 2:

\[ I_{xx_2} = I_{zy_2} = \frac{MR^2}{2} \]

\[ I_{yy_2} = MR^2 \]

Then,

\[ I_{xx} = MR^2 \quad \text{and} \quad I = I_1 + I_2 \]

\[ I_{yy} = \frac{3}{2} MR^2 \]

\[ I_{zz} = \frac{3}{2} MR^2 \]

all others are zero.

Principal axes: \( \hat{x}, \hat{y} \) and \( \hat{z} \).

By symmetry, we get an acceptable answer.
\[
\begin{align*}
\frac{dL}{dt} + \hat{\omega} \times \hat{L} &= \hat{N} \\
L_i &= \mathcal{K}_{ij} \\nI &= \begin{pmatrix}
1 & 0 & 0 \\
0 & \frac{3}{2} & 0 \\
0 & 0 & \frac{3}{2}
\end{pmatrix} M R^2 \\
\hat{L} &= \begin{pmatrix}
1 \\
\frac{3}{2} \\
\frac{3}{2}
\end{pmatrix} \begin{pmatrix}
\frac{1}{\sqrt{3}} M R^2 \omega_0 \\
0
\end{pmatrix} \\
&= \begin{pmatrix}
\frac{1}{\sqrt{3}} M R^2 \omega_0 \\
0
\end{pmatrix} \\
\hat{N} &\text{ is fixed such that } \frac{dL}{dt} = 0 \\
\Rightarrow \hat{N} &= \hat{\omega} \times \hat{L} \\
&= \begin{pmatrix}
\hat{\omega} & \hat{L} & \hat{N} \end{pmatrix} \\
&= \begin{pmatrix}
\hat{\omega} & \begin{pmatrix}
\frac{1}{\sqrt{3}} M R^2 \omega_0 \\
0
\end{pmatrix} & \hat{N}_N \end{pmatrix} \\
\hat{N} &= \frac{M R^2 \omega_0^2}{\sqrt{3}} \begin{pmatrix}
0 \\
0 \\
\frac{3}{2} - 1
\end{pmatrix}
\end{align*}
\]
1) For no net torque,
\[ \frac{d\vec{L}}{dt} + \vec{\omega} \times \vec{L} = 0 \]

And

\[ \vec{\omega} \times \vec{L} = \omega_x \vec{T} \]

\[ L_x = \omega_x I_x, \quad \omega_x = \omega \]

Then

\[ \begin{align*}
\dot{I}_x &= \omega_y I_z - \omega_z I_y = 0 \\
\dot{I}_y &= \omega_x I_z - \omega_z \omega_y I_y = 0 \\
\dot{I}_z &= \omega_y \omega_z (I_z - I_y) = 0 \\
\dot{\omega}_y &= \omega_x \omega_z (I_z - I_x) = 0 \\
\dot{\omega}_z &= \omega_y (I_y - I_x) = 0
\end{align*} \]

For
\[ I_x = mR^2 \]
\[ I_y = \frac{3}{2} mR^2 \]
\[ I_z = \frac{3}{2} mR^2 \]

\[ \dot{\omega}_x + \omega_y \omega_z = \dot{\omega}_x = 0 \quad (I_z - I_y = 0) \]
\[ \omega_y \left( \frac{3}{2} \right) + \omega_x \omega_z \left( -\frac{1}{3} \right) = 0 \]
\[ \omega_z \left( \frac{3}{2} \right) + \omega_x \omega_y \left( \frac{1}{2} \right) = 0 \]
\[ L = L_0 \frac{\ddot{X}}{I_x \omega_{x0}^2} \]

At \( t = 0 \),
\[ \dot{L} = L_0 \frac{\ddot{X}}{I_x \omega_{x0}^2} = I_x \omega_{x0}^2 / L_0 \]

\[ \Rightarrow \omega_{x0} = \frac{L_0}{\frac{I_x}{2} \omega_{x0}^2} \]

\[ \omega_x = 0 \Rightarrow \omega_x \text{ remain constant} \]

\[ 3 \omega_y 0 = 0 \]
\[ 3 \ddot{\omega}_2 + \omega_y \omega_{x0} = 0 \]
\[ \Rightarrow 3 \ddot{\omega}_2 + \omega_y \omega_{x0} = 0 \]

\[ \omega_2 (0) = 0 \]
\[ \omega_y (0) = \frac{L_0}{\frac{I_x}{2} \omega_{x0}^2} \]

\[ \Rightarrow \omega_y (t) = \omega_y (0) \cos \left( \frac{\omega_{x0} t}{3} \right) \]

and

\[ \Rightarrow 3 \left( \frac{-3 \ddot{\omega}_2}{\omega_{x0}} \right) - \omega_2 \omega_{x0} = 0 \]
\[ \ddot{\omega}_2 + \frac{\omega_x^2}{\omega_{x0}^2} \omega_2 = 0 \]

\[ \Rightarrow \omega_y + \frac{\omega_x^2}{\omega_{x0}^2} \omega_y = 0 \Rightarrow \omega_y (t) = \frac{L_0}{\frac{I_x}{2} \omega_{x0}^2} \cos \left( \frac{\omega_{x0} t}{3} \right) \]

\[ \omega_2 (t) = \frac{L_0}{\frac{I_x}{2} \omega_{x0}^2} \sin \left( \frac{\omega_{x0} t}{3} \right) \]

\( \omega_x \) precesses around \( \omega_y \) with frequency

\[ \omega_x = \frac{L_0}{\frac{I_x}{2} \omega_{x0}^2} \]
\[ Skn \bar{N} - 4 \text{ Relativity} \]

\( (a) \quad E^2 = p^2 c^2 = m^2 c^4 \)

\[ E^2 = m^2 c^4 \]
\[ E^2 = c^2 \sqrt{m^2 c^4 + p^2} \]
\[ E \cdot c = |p| c \]

\[ m^2 c^2 = c^2 \sqrt{m^2 c^4 + p^2} + |p| c \]

\[ (m^2 c^2 - |p| c)^2 = m^2 c^2 + p^2 \]

\[ |p| c = \frac{m^2 c^2 - m^2}{2m} = \gamma m c \]

\( (b) \quad p^2 = \frac{m^2 + m^2}{2m} = m^2 c^2 \)

\[ \frac{p}{E} = \frac{\gamma c}{c} \quad \frac{E}{\gamma c^2} = \frac{m^2 c^2 - m^2}{m^2 c^2 + m^2 c^2} \]