Expected Utility and the Density Matrix

Prashanth S. Venkataram

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This paper aims to analyze the effects of adding quantum mechanics to the expected utility problem. That is, how is an agent’s behavior regarding taking a gamble among many states of the world modified when those states fall in the framework of quantum mechanics? The paper restricts analysis to the two-state system for simplicity. Expected utility and the classical results are reviewed. Quantum mechanics and superposition are brought into the picture, followed by the introduction of more general classical-esque state mixtures through density matrices. Finally, the expansion of possibilities through the addition of quantum entanglement is discussed. This paper assumes moderate familiarity of the principles and formalism of single-particle quantum mechanics as well as an ability to pick up on basic understanding of density matrices, but does not assume too much familiarity with consumer choice theory.

1 Expected Utility

Agents make choices over bundles of goods. If these choices satisfy what is known as the weak axiom of revealed preferences, then these choices can be represented by the maximization of a complete and transitive preference. Typically, if the differences between bundles lie in the different quantities of the goods involved, then the ordering of preferences can be mapped to relations between real numbers, and this mapping of preferences to real numbers gives rise to utility. Utility is a function of a bundle that outputs a real number such that preference relations between bundles are preserved as real number relations between their utilities. As preferences are ordinal rankings, utility is a strictly ordinal rank; utility functions that are related by any increasing transformation represent preferences that cannot be distinguished. The maximization of complete and transitive preferences can be turned into a problem of maximizing utility, and the maximum utility as a function of the agent’s wealth and the prices of the bundle goods, which are exogenous variables, is known as the indirect utility. For the purposes of discussing expected utility, the prices can be discarded, leaving only wealth as the determinant of indirect utility.

Interestingly, although (direct) utility is strictly ordinal, indirect utility does have some cardinal meaning, in that the curvature of the indirect utility as a function of wealth gives the attitude of the agent towards taking risks for different levels of wealth. Because agents prefer more of all goods to less, agents prefer more wealth to less, so if the indirect utility is given by $V = V(w)$ for wealth $w$, then $\partial_w V(w) > 0$ for all $w$. However, the second derivative, which gives the curvature, depends on the person. Someone who is risk-neutral is defined to have $\partial^2_w V(w) = 0$. Someone who is risk-averse is defined to have $\partial^2_w V(w) < 0$, and the opposite is true for someone who is risk-loving. Furthermore, risk aversion can be quantified by the measure of absolute risk aversion, defined to be $-\frac{\partial^2 V(w)}{\partial w V(w)}$, while the measure of relative risk aversion is the measure of absolute risk aversion $|\psi\rangle$ multiplied by $w$.

A gamble will have an expected payoff

$$E(w) = \sum_{\psi} w(\psi)p(\psi)$$

(1)

given states $\psi$ having payoffs $w(\psi)$ and probabilities $p(\psi)$. Many people tend to avoid such gambles even if $E(w) > 0$. If people simply made decisions based on wealth, this would be considered irrational. However, Von Neumann and Morgenstern showed instead that people tend to maximize their expected utility defined by

$$E(V) = \sum_{\psi} V(w(\psi))p(\psi)$$

(2)

and maximization of this complete and transitive preference is not incompatible with risk aversion at all.

For simplicity, the only systems that will be considered will be two-state systems. Furthermore, the formula for expected utility will hold regardless of what the probabilities are, as long as the probabilities themselves follow the usual rules. Therefore, the rest of the paper will be devoted to exploring the different sources and forms of these probability functions.
2 Classical Probabilities

A typical two-state classical system for the analysis of expected utility would be in the flipping of a coin. The coin can either land heads or tails. Therefore, the probabilities are simply those of getting heads \( p_1 \equiv p \) or tails \( p_1 \equiv 1 - p \), so the expected utility becomes

\[
\mathbb{E}(V) = pV(w_1) + (1 - p)V(w_2)
\]

where \( p \) does not necessarily have to be \( \frac{1}{2} \), allowing for a biased coin for generality. Furthermore, if two people are playing, the two coins are typically independent of each other, so the choices of each person are determined in identical ways even if \( V(w) \) is not the same for each person.

3 Quantum Superpositions

The two-state system commonly used in quantum mechanics is that of the electron spin. Classically, a particle is described by a spin vector \( \vec{\sigma} = \sigma_x \hat{e}_x + \sigma_y \hat{e}_y + \sigma_z \hat{e}_z \) that may take any magnitude and point in any direction. This fully describes the spin state of the electron.

Quantum mechanically, the spin state is no longer a Cartesian vector \( \vec{\sigma} \) with definite magnitude and direction but is instead described by an abstract vector \( |\psi\rangle \) in a two-dimensional complex Hilbert space. The reason why the space is two-dimensional is because the electron can only take one of two possible values (\( \pm 1 \)) for spin measured in a particular direction (whose eigenstates would be \( |\uparrow\rangle \) or \( |\downarrow\rangle \)). Furthermore, it is no longer possible to measure spin in more than one direction without uncertainty. Mathematically, this means that the components of spin \( \sigma_j \) for \( j \in \{x,y,z\} \) are now noncommuting operators on that Hilbert space, and states of definite spin in a given direction are eigenstates of that spin operator in the Hilbert space; defining the commutator of two operators as \([A,B] = AB - BA\), then \( [\sigma_j, \sigma_k] = 2i \sum_{l \in \{x,y,z\}} \epsilon_{jkl} \sigma_l \neq 0 \) implies that two components of spin cannot be diagonalized simultaneously and therefore cannot have simultaneous eigenstates.

What else does this imply? This implies that eigenstates of spin in a particular direction are superpositions of spin eigenstates in other directions. For a specific example, denoting the eigenstates of \( \sigma_j \) through the relations \( \sigma_j |\uparrow_j\rangle = |\uparrow_j\rangle \) and \( \sigma_j |\downarrow_j\rangle = -|\downarrow_j\rangle \), and working in the basis \( \{ |\uparrow_z\rangle, |\downarrow_z\rangle \} \) for convenience, then

\[
|\uparrow_x\rangle = \frac{1}{\sqrt{2}} (|\uparrow_z\rangle + |\downarrow_z\rangle) \tag{4}
\]

\[
|\downarrow_x\rangle = \frac{1}{\sqrt{2}} (|\uparrow_z\rangle - |\downarrow_z\rangle) \tag{5}
\]

\[
|\uparrow_y\rangle = \frac{1}{\sqrt{2}} (|\uparrow_z\rangle + i|\downarrow_z\rangle) \tag{6}
\]

\[
|\downarrow_y\rangle = \frac{1}{\sqrt{2}} (|\uparrow_z\rangle - i|\downarrow_z\rangle). \tag{7}
\]

To make the superposition relations more clear, the eigenstates \( \{ |\uparrow_z\rangle, |\downarrow_z\rangle \} \) can be written in terms of their counterparts along the \( x \)- and \( y \)-directions:

\[
|\uparrow_z\rangle = \frac{1}{\sqrt{2}} (|\uparrow_x\rangle + |\downarrow_x\rangle) = \frac{1}{\sqrt{2}} (|\uparrow_y\rangle + |\downarrow_y\rangle) \tag{8}
\]

\[
|\downarrow_z\rangle = \frac{1}{\sqrt{2}} (|\uparrow_x\rangle - |\downarrow_x\rangle) = \frac{i}{\sqrt{2}} (|\uparrow_y\rangle - |\downarrow_y\rangle). \tag{9}
\]

What does this mean now? Let us suppose that a large system of particles is prepared such that all of them are in the \( |\uparrow_x\rangle \) state. If this is the case, then if a measurement of \( S_z \) is performed on all of them, the measured value will be \( +\frac{h}{2} \) with certainty. By contrast, if a measurement of \( S_z \) is performed, then the measured values will be one of \( \pm \frac{h}{2} \) with equal probabilities (and the same holds for \( S_y \)). After a measurement of \( S_z \) (or \( S_y \)) is performed, if \( S_z \) is measured again, then as each particle state has collapsed into an eigenstate of \( S_z \) (or \( S_y \)), then each measurement of \( S_x \) will yield \( +\frac{h}{2} \) corresponding to the original state only with half the probability.

This has profound implications for the question of expected utility. Previously, the person making the payout (hereafter called the game owner) in a game could only measure one of two possibilities for the coin flip (heads or tails) and therefore would be assured of the probabilities and payouts for those outcomes. Now, however, the game owner has a choice of which direction to measure spin and how much to pay for up versus down in any given...
direction. For example, a state prepared as \( |\psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow_z\rangle + |\downarrow_z\rangle) \) will be measured to have the eigenvalue \( \sigma_z = +1 \) with probability \( \frac{1}{2} \) and the eigenvalue \( \sigma_z = -1 \) with probability \( \frac{1}{2} \); however, the state is the same as \( |\psi\rangle = |\uparrow_x\rangle \), which will yield the eigenvalue \( \sigma_x = +1 \) with certainty if spin along the \( x \)- rather than \( z \)-axis is measured.

In general, spin along any direction given by the normal vector \( \vec{e}(\theta, \varphi) = \sin(\theta) \cos(\varphi) e_x + \sin(\theta) \sin(\varphi) e_y + \cos(\theta) e_z \) can be measured as \( \sigma(\theta, \varphi) = \sin(\theta) \cos(\varphi) \sigma_x + \sin(\theta) \sin(\varphi) \sigma_y + \cos(\theta) \sigma_z \), and its eigenstates are given in the eigenbasis of \( \sigma_z \) by convention as

\[
| \uparrow (\theta, \varphi) \rangle = \cos \left( \frac{\theta}{2} \right) | \uparrow_z \rangle + \sin \left( \frac{\theta}{2} \right) e^{i\varphi} | \downarrow_z \rangle \tag{10}
\]

\[
| \downarrow (\theta, \varphi) \rangle = -\sin \left( \frac{\theta}{2} \right) e^{-i\varphi} | \uparrow_z \rangle + \cos \left( \frac{\theta}{2} \right) | \downarrow_z \rangle. \tag{11}
\]

Because any arbitrary state is given as \( |\psi\rangle = \alpha |\uparrow_z\rangle + \beta |\downarrow_z\rangle \) satisfying \((\alpha, \beta) \in \{C^2 : |\alpha|^2 + |\beta|^2 = 1\} \), then \( \alpha \) and \( \beta \) can be chosen such that \( |\psi\rangle = | \uparrow (\theta_0, \varphi_0) \rangle \) (or alternatively \( | \downarrow (\theta_0, \varphi_0) \rangle \)) for some particular \((\theta_0, \varphi_0)\). Hence, the agent now has to worry about not just whether the outcome will be up or down but also in which direction of spin the state will be measured to produce an outcome. The agent may believe that the game owner will measure in a direction \((\theta, \varphi)\) with (classical) probability density \( q(\theta, \varphi) \) satisfying \( \int q(\theta, \varphi) d\theta d\varphi = 1 \) as is proper for a probability density; this quantity is nonzero only for \( (\theta, \varphi) \in \{0 \leq \theta < \frac{\pi}{2}, 0 \leq \varphi < 2\pi\} \cup \{\theta = \frac{\pi}{2}, 0 \leq \varphi < \pi\} \) because spin up in a given Cartesian direction is spin down in the opposite Cartesian direction, so this set of \((\theta, \varphi)\) is the smallest (though not uniquely so) such choice which will give nonredundant sets of orthogonal states \(| \uparrow (\theta, \varphi) \rangle \) & \( | \downarrow (\theta, \varphi) \rangle \). If the game owner prepares the system in the state \(|\psi\rangle = \alpha |\uparrow_z\rangle + \beta |\downarrow_z\rangle \) satisfying \((\alpha, \beta) \in \{C^2 : |\alpha|^2 + |\beta|^2 = 1\} \), and if the direction \((\theta, \varphi)\) is chosen for a measurement, then the probability of measuring up in that direction is given by the inner product \(|\langle \uparrow (\theta, \varphi) | \psi \rangle|^2 \) (and the analogue for down in that direction). Hence, as the payoff in up or down states needs to be defined for each direction, then weighting over all directions yields the expected utility

\[
E(V) = \int \int q(\theta, \varphi) (|\langle \uparrow (\theta, \varphi) | \psi \rangle|^2 V(w_\uparrow(\theta, \varphi)) + |\langle \downarrow (\theta, \varphi) | \psi \rangle|^2 V(w_\downarrow(\theta, \varphi))) d\theta d\varphi. \tag{12}
\]

For example, let us consider \(|\psi\rangle = \sqrt{\frac{2}{3}} | \uparrow_z \rangle + \sqrt{\frac{1}{3}} | \downarrow_z \rangle\), which might classically correspond to a weighted coin that will only get heads with probability \( \frac{2}{3} \). Let us also consider the measurement probability density \( q(\theta, \varphi) = \frac{1}{6} \delta(\theta) + \delta \left( \theta - \frac{\pi}{2} \right) \left( \frac{1}{2} \delta(\varphi) + \frac{1}{2} \delta(\varphi - \pi) \right) \), which corresponds to measuring along \( z \) with probability \( \frac{1}{6} \), measuring along \( x \) with probability \( \frac{1}{2} \), and measuring along \( y \) with probability \( \frac{1}{3} \). The amplitude of measuring \(|\psi\rangle\) to be an eigenstate either up or down in a given direction is \(|\langle \uparrow (\theta, \varphi) | \psi \rangle| = \sqrt{\frac{2}{3}} \cos(\theta) + \sqrt{\frac{1}{3}} \sin(\theta) e^{i\varphi} \) or \(|\langle \downarrow (\theta, \varphi) | \psi \rangle| = -\sqrt{\frac{2}{3}} \sin(\theta) + \sqrt{\frac{1}{3}} \cos(\theta) e^{i\varphi} \), respectively. Specifically, these respective amplitudes are \( \sqrt{\frac{2}{3}} \) or \(-\sqrt{\frac{2}{3}}\) along \( x \), \(-\sqrt{\frac{1}{3}}\) or \(\sqrt{\frac{1}{3}}\) along \( y \), or \(\sqrt{\frac{2}{3}}\) or \(-\sqrt{\frac{2}{3}}\) along \( z \). Hence, the expected utility in this case is

\[
E(V) = \frac{1}{6} \left( \frac{3}{5} V(w_\uparrow_z) + \frac{2}{5} V(w_\downarrow_z) \right) \tag{13}
\]

\[
+ \frac{1}{3} \left( \frac{3}{5} V(w_\uparrow_y) + \frac{2}{5} V(w_\downarrow_y) \right) \tag{14}
\]

\[
+ \frac{1}{2} \left( \frac{2}{5} V(w_\uparrow_x) + \frac{3}{5} V(w_\downarrow_x) \right). \tag{15}
\]

### 4 Density Matrices

The above only holds true if the agent is sure that the game owner has prepared the electron in a pure state \(|\psi\rangle\) (but doesn’t know what that state is). Generally, the agent may be unsure of the initial state of the electron as well, and may assign probabilities for the electron being prepared in one state versus another. Such probabilities can be associated with their states through the density matrix. A general density matrix can be represented as a sum of arbitrary, not necessarily orthogonal states

\[
\rho = \sum_j p_j |\psi_j\rangle \langle \psi_j | \tag{16}
\]
where the probabilities \( p_j \) sum to unity. In this particular case, as the Hilbert space is two-dimensional, the density matrix is a 2 by 2 matrix when represented in the eigenbasis of a given \( \sigma(\theta, \varphi) \). Generally, for this two-state system, the density matrix is a statistical mixture of up and down states in all directions given by

\[
\rho = \int \int (p_\uparrow(\theta, \varphi) | \uparrow(\theta, \varphi) \rangle \langle \uparrow(\theta, \varphi)| + p_\downarrow(\theta, \varphi)| \downarrow(\theta, \varphi) \rangle \langle \downarrow(\theta, \varphi)|) \, d\theta \, d\varphi
\]

where the probability densities of the up and down state mixtures are defined only for \( (\theta, \varphi) \in \{0 \leq \theta < \frac{\pi}{2}, 0 \leq \varphi < 2\pi\} \cup \{\theta = \frac{\pi}{2}, 0 \leq \varphi < \pi\} \) and satisfy \( \int \int (p_\uparrow(\theta, \varphi) + p_\downarrow(\theta, \varphi)) \, d\theta \, d\varphi = 1 \). The interpretation of this is that the game owner may have a large number of electrons whose state is up or down in a particular direction according to \( \theta, \varphi \leq \frac{\pi}{2} \) or \( \theta = \frac{\pi}{2}, 0 \leq \varphi < \pi \), respectively.

To go back to the pure state case, the corresponding density matrix is simply \( \rho = |\psi\rangle \langle \psi| \), which means that \( |\langle \phi | \psi \rangle|^2 = |\langle \phi | \rho | \phi \rangle| \) for any state \( |\phi\rangle \). Hence, the expression for expected utility even in the more general mixed state case becomes

\[
\mathbb{E}(V) = \int \int q(\theta, \varphi)(|\uparrow(\theta, \varphi) \rangle \langle \uparrow(\theta, \varphi)| + |\downarrow(\theta, \varphi) \rangle \langle \downarrow(\theta, \varphi)|) \, d\theta \, d\varphi
\]

which now accounts for the fact that the state of the system which may be measured to be up or down in a given direction is a statistical mixture of states in all directions.

## 5 Entanglement

In the classical game, if two people played, their outcomes wouldn’t affect each other in the slightest. This is not true in the quantum version of the game, where the individual states of the two electrons are combined into a single state existing in the direct product of the two individual Hilbert spaces. (Technically, as electrons are fermions, only symmetric or antisymmetric entanglements are allowed, but this technical issue can be rectified by discussing the spin entanglement of two distinguishable spin-\( \frac{1}{2} \) particles, like an electron and a muon.) A general state is now given by

\[
|\psi\rangle = \alpha_1|\uparrow_z, \uparrow_z\rangle + \alpha_2|\uparrow_z, \downarrow_z\rangle + \alpha_3|\downarrow_z, \uparrow_z\rangle + \alpha_4|\downarrow_z, \downarrow_z\rangle
\]

for \( \alpha_j \in \mathbb{C} : \sum_j |\alpha_j|^2 = 1 \) for \( j \in \{1, 2, 3, 4\} \), where the tensor product of states is collapsed into the notation \( |\uparrow_z, \uparrow_z\rangle \equiv |\uparrow_z\rangle \otimes |\uparrow_z\rangle \) and means that the first and second electrons are both in up states along the \( z \)-axis (and likewise for the other permutations of up and down). Sometimes, \( |\psi\rangle \) can be separated into the tensor product of two overall states, in which case it is called a separable state, for which measuring the state of one electron does not affect the state of the other electron. More generally, though, \( |\psi\rangle \) cannot be separated like that and is an entangled state, where there exist fundamental correlations between measurements of the states of the two electrons and for which there is no classical analogue.

For simplicity, let us assume that only \( \sigma_z \) is measured for each electron. Let us also assume that the payoffs \( w_\uparrow_z \) and \( w_\downarrow_z \) are the same for both players. Let us label the player whose electron is the first in the tensor product as ‘A’ and the second as ‘B’ with indirect utility functions \( V_A(w) \) and \( V_B(w) \). If the state of the first electron is measured first, player ‘A’ has a probability given by \( |\alpha_1|^2 + |\alpha_2|^2 \) of getting the electron up, and a probability of \( |\alpha_3|^2 + |\alpha_4|^2 \) of getting the electron down.

If the former possibility occurs, then the state of the second electron collapses into \( |\psi_B\rangle = (|\alpha_1|^2 + |\alpha_2|^2)^{-\frac{1}{2}} (\alpha_1|\uparrow_z\rangle + \alpha_2|\downarrow_z\rangle) \), and may be measured to be up with probability \( \frac{|\alpha_1|^2}{|\alpha_1|^2 + |\alpha_2|^2} \) or down with probability \( \frac{|\alpha_2|^2}{|\alpha_1|^2 + |\alpha_2|^2} \). If the latter possibility occurs, then the state of the second electron likewise collapses into \( |\psi_B\rangle = (|\alpha_3|^2 + |\alpha_4|^2)^{-\frac{1}{2}} (\alpha_3|\uparrow_z\rangle + \alpha_4|\downarrow_z\rangle) \), and may be measured to be up with probability \( \frac{|\alpha_3|^2}{|\alpha_3|^2 + |\alpha_4|^2} \) or down with probability \( \frac{|\alpha_4|^2}{|\alpha_3|^2 + |\alpha_4|^2} \).

Now if instead the second electron is measured first, it may be measured as up with probability \( |\alpha_1|^2 + |\alpha_3|^2 \) or down with probability \( |\alpha_2|^2 + |\alpha_4|^2 \). If the former possibility occurs, then the state of the first electron collapses into \( |\psi_A\rangle = (|\alpha_1|^2 + |\alpha_3|^2)^{-\frac{1}{2}} (\alpha_1|\uparrow_z\rangle + \alpha_3|\downarrow_z\rangle) \), and may be measured to be up with probability \( \frac{|\alpha_1|^2}{|\alpha_1|^2 + |\alpha_3|^2} \) or down with probability \( \frac{|\alpha_3|^2}{|\alpha_1|^2 + |\alpha_3|^2} \). If the latter possibility occurs, then the state of the first electron likewise collapses into \( |\psi_A\rangle = (|\alpha_2|^2 + |\alpha_4|^2)^{-\frac{1}{2}} (\alpha_2|\uparrow_z\rangle + \alpha_4|\downarrow_z\rangle) \), and may be measured to be up with probability \( \frac{|\alpha_2|^2}{|\alpha_2|^2 + |\alpha_4|^2} \) or down with probability \( \frac{|\alpha_4|^2}{|\alpha_2|^2 + |\alpha_4|^2} \).
Finally, let us say that the players agree that the probability of player \( l \in \{ A, B \} \) going first is \( p_l \). Putting these all together, the expected utility of player ‘A’ is

\[
E(V_A) = (|\alpha_1|^2 + |\alpha_2|^2)V_A(w_{\uparrow z}) + (|\alpha_3|^2 + |\alpha_4|^2)V_A(w_{\downarrow z})
\]

(20)

and of ‘B’ is

\[
E(V_B) = (|\alpha_1|^2 + |\alpha_3|^2)V_B(w_{\uparrow z}) + (|\alpha_2|^2 + |\alpha_4|^2)V_B(w_{\downarrow z})
\]

(21)

which is independent of \( p_l \). The nice thing about this result is that it preserves some level of independence between the players in that whose state is measured first is irrelevant. That said, the key point remains that quantum correlations which have no classical analogue allow the expected utilities of the different players to tie together except in the special cases of separable states.

For example, \( |\psi\rangle = \sqrt{\frac{1}{12}} |\uparrow_z, \uparrow_z\rangle + \sqrt{\frac{1}{6}} |\uparrow_z, \downarrow_z\rangle + \sqrt{\frac{1}{2}} |\downarrow_z, \uparrow_z\rangle + \frac{1}{2} |\downarrow_z, \downarrow_z\rangle \), then the expected utility of player ‘A’ is

\[
E(V_A) = \frac{1}{4} V_A(w_{\uparrow z}) + \frac{3}{4} V_A(w_{\downarrow z})
\]

(22)

and of ‘B’ is

\[
E(V_B) = \frac{7}{12} V_B(w_{\uparrow z}) + \frac{5}{12} V_B(w_{\downarrow z}).
\]

(23)