Electrons in conductors at finite nonzero temperatures produce measurable voltages due to thermal noise, even if the conductor is not driven by any other external voltage. The thermal noise voltage, called Johnson noise, is posited to depend only on the frequency width of the signal, as well as the temperature and resistance of the material. From the measurement of Johnson noise voltages for resistors of many different resistances at the same temperature, the proportionality constant, which is the Boltzmann constant, can be found. In addition, from the measurement of Johnson noise voltages for a single resistor at many different temperatures, the Boltzmann constant as well as the absolute zero temperature can be found.

Electrons in conductors are discrete particles rather than a continuous fluid of charges. This means that in addition to Johnson noise, electrons produce fluctuations in any current passing through a conductor solely because they are discrete charges rather than a continuous charge density; these fluctuations constitute shot noise. Measuring unamplified and amplified voltages from a specialized circuit allows for the determination of the value of the electron charge.

In 1913, Robert A. Millikan improved [1] upon the oil drop experimental setup to measure the charge of the electron as well as the Avogadro constant. The experimental apparatus was very sensitive, as the charged oil drop was assumed to remain at the same position in the applied electric field for several hours, whereas it could have partially or fully evaporated during that time. An alternate way for deriving the electron charge is through the measurement of shot noise, which is the fluctuation in any current passed through a conductor exactly because of the discrete charge-carrying nature of the electron. Passing current through a conductor for a given amount of time gives a mean value for that current along with a square mean value, and the relation of the latter to the former arises through the electron charge along with the amount of time that the current was applied. Hence, through appropriate amplification of the currents and their fluctuations, the electron charge can be measured as a microscopic quantity arising from macroscopic effects.

While the shot noise experiment does not also measure the Avogadro constant like the oil drop experiment does, the Boltzmann constant is related to the molar ideal gas constant exactly through the Avogadro constant, and as the molar ideal gas constant has been measured to a high degree of precision and accuracy, determination of the Boltzmann constant also yields the Avogadro constant.

At nonzero finite temperatures, electrons move due to thermal excitations, and this motion produces a nonzero fluctuating thermal noise voltage called Johnson noise. Although the molar ideal gas constant multiplied by the number of moles of the gas is equal to the product of the number of particles in the gas multiplied by the Boltzmann constant, as determined by considering the gas particle degrees of freedom, the latter two quantities cannot be separated if the particle number cannot be determined. However, the Johnson noise voltage squared has a dependence on the resistance and temperature of a conductor similar to the dependence of the pressure of an ideal gas on its particle number and temperature. Furthermore, the Johnson noise voltage only depends on those macroscopic properties with the proportionality constant being the Boltzmann constant independent of any extensive quantity like particle number.

I. SHOT NOISE

I.1. Theoretical Basis

The fluctuations in a current across a conductor due to shot noise can be decomposed into its Fourier components: \( I(t) = \sum_k I_k(t) \). The mean square current measured over a time \( t \) is \( \langle I^2 \rangle = \frac{1}{2} \int_0^\infty I^2(f')df' \). Considering that the allowed frequencies for the Fourier components are very closely spaced compared to the frequencies themselves [2], summing over the phases of the Fourier components and transforming back to a sum over frequencies within a continuous infinitesimal frequency interval yields the differential mean square current contribution \( d\langle I^2 \rangle = 2q_e \langle I \rangle df \) for a frequency interval \( df \) and a mean current \( \langle I \rangle \). Accounting for gains in the system, the squared AC voltage for shot noise passing through a resistor \( R \) and then being amplified is \( V_{AC}^2 = 2q_e RV_{DC} \int_0^\infty (g(f))^2df + V_{other}^2 \), where \( g(f) \) is the gain of the system, \( V_{DC} = R\langle I \rangle \) is the Fourier amplitude of the voltage at \( f = 0 \), and \( V_{other} \) accounts for other sources of noise like Johnson noise.

I.2. Apparatus and Method

Before any shot noise measurements can be performed, calibration must be performed to determine the dependence of the gain in the full circuit on the frequency stem-
FIG. 1. Schematics of the calibration (left) and measurement (right) apparatuses for shot noise (adapted from [2]).

ming from the filtering by the preamplifier and the bandpass filter. As seen in figure 1 (on the left), a 20 mV RMS AC signal is sent from an Agilent function generator into the input of the photodiode box, which has all other switches (except for the main power) turned off for this step. The stage 1 output of the photodiode is capped, while the stage 2 output, which is amplified by a factor of 10, is sent into an SRS SR560 preamplifier that effects a further factor of 10 amplification. This amplified signal, now two orders of magnitude higher in amplitude than the input, is sent through a Krohn-Hite bandpass filter that removes low- and high-frequency signals. The resulting signal RMS voltage is then viewed on an Agilent multimeter.

Each pair of input and output voltages was measured once for each frequency using the single measurement tool after the average fluctuation was determined. The resulting gain curve is shown in figure 2. The point at 120 kHz was the highest-frequency point taken. The line of square AC voltage versus DC voltage has in its coefficient the integral of the square of the gain over all frequencies; this means that the gain curve must approach 0 fast enough. For the discrete sum approximation to the integral, the way to ensure this was to subtract the output voltage at 120 kHz, which was found to be essentially noise, from all the other output voltages in quadrature and then recalculate the gain. This would ensure that the sum of the gain squared over frequency would converge.

After the calibration, the shot noise procedure can be performed. Now, the input is capped, as the source of shot noise consists of electrons being ejected by a light inside the photodiode box illuminating an electrode through the photoelectric effect. These electrons pass through a specialized circuit where a DC voltage across a resistor in that circuit of known \( R = 475 \, k\Omega \) is sent through the stage 1 output into another multimeter. The AC voltage passes through amplification by a factor of 10 and the rest of the setup similar to what happened in the gain calibration procedure. Eleven different mean voltages were measured, and each mean voltage consisted of 10 to 12 AC and DC voltage measurements.

I.3. Results

As shown in figure 3, the squared AC voltage output after stage 2 versus the DC voltage output after stage 1 from the photodiode box fits to a line; in this case, the DC voltage output has been multiplied by the resistance of 475 k\( \Omega \) and the integral of the square of the gain curve over all frequencies. A linear fit produces the slope and the vertical intercept, the slope in this case being the electron charge and the intercept being the noisy AC voltage present even when no DC voltage can be produced. The linear fitting algorithm used cannot take errors on the horizontal values initially, so an initial fit was performed only considering the errors on the squared AC voltages, and then further fits were performed iteratively propagating the errors on the horizontal terms with the new slopes and adding them in quadrature with the vertical errors to produce overall vertical errors. This process was repeated until the fit parameter values and errors stabilized. From this, the electron charge was found to be \((1.897 \pm 0.023) \times 10^{-19} \text{C}\), which differs from the accepted value of \(1.602 \times 10^{-19} \text{C}\) by twelve times the uncertainty in the value measured in this experiment. The squared noise voltage is \((1.0676 \pm 0.0020) \times 10^{-4} \text{V}^2\). With 9 degrees of freedom, the reduced \(\chi^2 = 3.18\).
II. JOHNSON NOISE

II.1. Theoretical Basis

In 1928, Johnson showed that the mean square of the voltage across a conductor not driven by any other voltage is proportional only to the resistance and temperature of the conductor. Nyquist [3] elaborated on this by considering the thermal energy of the electromagnetic field passed through a transmission line shorted at both ends[2]. As each mode contributes $kT$ to the thermal energy, where $T$ is the absolute temperature and $k$ is the Boltzmann proportionality constant to be found, the differential squared voltage over an interval of frequency for noisy signals is $dV^2 = 4kRTdf$. However, the equivalence of the shorted transmission line problem to that of finding the thermal energies of an electromagnetic field around an arbitrary conductor requires a capacitance as well. Including any gain, the squared voltage across the conductor becomes $V^2 = 4kRT\int_0^\infty \frac{(g(f))^2}{1+(2\pi fRC)^2} df$. This can be fitted to a line for different resistors at a given temperature to yield $k$, or for a single resistor at different non-absolute temperatures to yield $k$ as well as the value of absolute zero temperature in that scale.

II.2. Apparatus and Method

FIG. 4. Schematics of the calibration (left) and measurement (right) apparatuses for shot noise (adapted from [2])

As with shot noise, before any Johnson noise measurements can be performed, calibration must be performed to determine the dependence of the gain in the full circuit on the frequency stemming from the filtering by the preamplifier and the bandpass filter. As seen in figure 4 (on the left), a 20 mV RMS AC signal is sent from an Agilent function generator through a Kay attenuator set at 26 dB attenuation. This splits into a direct output into one Agilent multimeter as well as another output into the SRS SR560 preamplifier. As with shot noise, the preamplifier outputs (though now with a gain of 1000) to the Krohn-Hite bandpass filter, which outputs to another Agilent multimeter.

Each pair of input and output voltages was measured once for each frequency using the single measurement tool after the average fluctuation was determined. The resulting gain curve is shown in figure 5. There was a point at 1 MHz that was taken on a different day from the other measurements, and that was the only measurement with a higher frequency than the one at 120 kHz. It provided a good estimate of the noise in the output voltage, but it was not included here as parts of the apparatus had changed from day to day. In Johnson noise analysis, the integral over the square of the gain curve over frequency is weighted by a term proportional to $f^{-2}$ for large $f$. This means that the gain curve does not have to approach zero as long as it does not grow faster than $f^2$. However, for consistency with the shot noise calibration and to ensure the reduction of noise from the output measurements, the output voltage at 1 MHz was subtracted in quadrature from all the output voltages, yielding the corrected gain curve shown here.

After the calibration, the Johnson noise procedure can be performed. For the resistor-dependent measurements, the temperature of the room and its fluctuations are measured once by a multimeter with a thermal couple connected. The resistance and fluctuations of each resistor is measured with a multimeter. Each resistor is put into position with a shielding aluminum beaker on top of the box, following which twenty measurements are made of the RMS AC noise voltage passing through the resistor, and ten measurements are made of the RMS AC noise voltage with the resistor shorted. The Johnson noise for nine resistors was measured like this.

For the temperature-dependent measurements, one resistor is picked out and its resistance and fluctuations measured by a multimeter. The temperature is measured first at room temperature, with fifteen noise voltage measurements made through the resistor and ten in the shorted circuit position. These temperature and noise voltage measurements were repeated for eight more temperatures, all coming from the Johnson noise box and attached resistor being inverted into an oven heating to approximately 150 degrees Celsius. This was again repeated for one final temperature, which was that of the boiling point of nitrogen, as the Johnson noise box was allowed to equilibrate with room temperature and then was once again inverted into a beaker of liquid nitrogen.

FIG. 5. Corrected gain curve for Johnson noise
II.3. Results

![Image](https://via.placeholder.com/150)

FIG. 6. $\chi^2$ versus $k$ and $C$: The cyan line indicates a 95% probability of $\chi^2$ inside that area, the yellow line similarly indicates 70%, the magenta line indicates the minimum $\chi^2$ plus one, and the red line the minimum $\chi^2$ plus two, where $\chi^2$ here is not reduced; lower values of $\chi^2$ are darker in shade.

Typically, the capacitance of the system is measured. However, our measurement was made on a different day, at which point the apparatus had changed significantly and irreversibly from our original setup, so the capacitance was taken to be a fit parameter for which $\chi^2$ should be minimized. At the value of $k$ crossing the minimum $\chi^2$, the value of capacitance, as determined in figure 6 by the level of $\chi^2$ increased from the minimum by one as per [4], the capacitance is $(6.108 \pm 0.040) \times 10^{-11}$ F, with a reduced $\chi^2 = 0.63$ for 7 degrees of freedom. The probability contours of 95% and 70% as representing two and one standard deviations, respectively, are not used as they are consistent with Gaussian statistics and not $\chi^2$ statistics.

As shown in figure 7, this capacitance and its error, along with the other parameters and their errors, can be used to minimize the variation (standard deviation) of $k$ divided by its mean as a function of resistance, because in this model, $k$ should be a constant independent of $R$. Calculating the means and standard deviations weighted by errors for each resistance yields $k = (1.4418\pm0.0083) \times 10^{-23}$ JK$^{-1}$, which is off by 7.5 standard deviations from the accepted $k = 1.38 \times 10^{-23}$ JK$^{-1}$.

As shown in figure 7, this capacitance is also used in the linear fit of $V^2$ versus $4kRTG$, where $G$ is the resistance- and capacitance-dependent integral of the square of the gain over frequency. From this, $k = (1.29 \pm 0.10) \times 10^{-23}$ JK$^{-1}$, while the absolute zero temperature in degrees Celsius is $T = -265 \pm 5$, which is off from the accepted $T = -273.15$ degrees Celsius by 1.6 times the uncertainty. The $\chi^2$ of this fit is 5.0. Thus, using the difference between the resistor-and temperature-dependent determinations of $k$ as a bound on its systematic error, the Boltzmann constant is $(1.44 \pm 0.00_{\text{stat}} pm 0.15_{\text{sys}}) \times 10^{-23}$ JK$^{-1}$.

![Image](https://via.placeholder.com/150)

FIG. 7. Top: $k$ as a function of $R$; Bottom: $V^2$ as a function of $T$.

III. CONCLUDING REMARKS

In shot noise, the electron charge was determined to differ from the accepted value by twelve times its uncertainty. In Johnson noise, the Boltzmann constant was determined to differ from the accepted value to within one times the uncertainty, while the absolute zero temperature was found to differ by 1.6 times its uncertainty.


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