Two random matrix central limit theorems

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References:


Available at my website or arXive
Two probabilistic models

- Non-intersecting random walks (Dyson’s Brownian motions, random Hexagon tiling)
- Last passage percolation

Universal fluctuations as in the random matrix theory at least in certain asymptotic regimes
Universality theorems in random matrix theory

A. Unitary invariant ensembles

The set of $N \times N$ Hermitian matrices $H$ with measure $e^{-N \text{tr} V(H)}$

Take $N \to \infty$.

Limiting density of eigenvalues depend on $V$.

But local statistics are independent of $V$: given by sine kernel (bulk) or Airy kernel (edge)

[Pastur-Shcherbina], [Bleher-Its], [Deift-Kricherbaum-McLaughlin-Venakides-Zhou], [Deift-Gioev]

$$
\mathbb{P}\left((\xi_{\max}(N) - a) b N^{2/3} \leq x \right) \to F_{TW}(x) = \det(1 - A |_{(x, \infty)})
$$

$$
\mathbb{P}\left(\text{no eigenvalue in } \left( x_0 - \frac{cx}{N}, x_0 + \frac{cx}{N} \right) \right) \to \det(1 - S |_{(-x, x)})
$$
\[ A(u, v) = \frac{\text{Ai}(u) \text{Ai}'(v) - \text{Ai}'(u) \text{Ai}(v)}{u - v} \]

\[ S(u, v) = \frac{\sin(\pi(u - v))}{\pi(u - v)} \]

**B. Orthogonal and Symplectic ensembles**

[Deift-Gioev]

**C. Wigner matrices**

Hermitian matrices with independent and ‘identically distributed’ entries

Limiting distribution of the largest eigenvalue is universal [Soshnikov]
Non-intersecting random walks

Example 1. Bernoulli walks

Picture by Propp, size=20

= random rhombus tiling of Hexagon
Propp, size=64
\( n \) = number of walks

\( T \) = time steps

For \( n = O(T) \to \infty \), limiting distributions are given by those of GUE

[Baik-Kriecherbauer-McLaughlin-Miller] (based on the algebraic work of [Johansson])
Example 2. Dyson process

$n$ Brownian bridge processes $B_t = (B^{(1)}_t, \ldots, B^{(n)}_t)$ for $t \in [0, 2]$ such that $B_0 = B_2 = 0$, conditioned not to intersect (i.e. $B^{(1)}_t > \cdots > B^{(n)}_t$).

Density of $B_t$ at time $t = 1$ is

$$c_n \cdot \prod_{1 \leq j < k \leq n} |b_j - b_k|^2 \prod_{j=1}^n e^{-b_j^2}$$

Eigenvalue density of $n \times n$ GUE!

Proof: Karlin-McGregor argument

$$\mathbb{P}(a_1 \rightarrow b_1, a_2 \rightarrow b_2, \text{no intersect})$$

$$= \mathbb{P}(a_1 \rightarrow b_1, a_2 \rightarrow b_2) - \mathbb{P}(a_1 \rightarrow b_2, a_2 \rightarrow b_1)$$

$$= \mathbb{P}(a_1 \rightarrow b_1)\mathbb{P}(a_2 \rightarrow b_2) - \mathbb{P}(a_1 \rightarrow b_2)\mathbb{P}(a_2 \rightarrow b_1)$$

$$= \det \left[ \mathbb{P}(a_j \rightarrow b_k) \right]_{1 \leq j, k \leq n}$$
Other examples.

Longest increasing subsequences, random Young diagram [Karlin]

Random domino tiling of Aztec diamond [Johansson]

A model for bus system in Cuernevaca, Mexico [Baik-Borodin-Deift-Suidan]

General non-intersecting random walks

Fix $n$, take $T \to \infty$: random walks $\approx$ Brownian motions.

If we take $T \to \infty$ first, and then $n \to \infty$, we obtain the random matrix limits.
Can we take $n, T \to \infty$ simultaneously?

Brownian approximation v.s. non-intersect conditioning

[Baik-Suidan] For general random variables with finite moment-generating function, yes when ‘$T \geq O(n^{8/3})$’.

- For a few special cases, yes even when $T = O(n)$
- True for general random variables? Open question.
Last passage percolation

i.i.d. random variables $X(i, j)$ at each site $(i, j) \in \mathbb{Z}^2$.

Admissible paths: $\Pi(N, k) = \text{the set of ‘up/right’ paths from } (1, 1) \text{ to } (N, k)$

Total $\binom{N+k-2}{N-1}$ paths

Last passage time:

$$L(N, k) := \max_{\pi \in \Pi(N, k)} \left\{ \sum_{(i, j) \in \pi} X(i, j) \right\}$$

Random growth model, queues in tandem, interacting particle systems
1. Expectation

(1) \( k = o(N), \lim_{N,k \to \infty} \frac{\mathbb{E}L(N,k) - \mu N}{\sqrt{Nk}} = 2\sigma \) [Seppääläinen]

(2) \( k = O(N), \lim_{n \to \infty} \frac{\mathbb{E}L([xn],[yn])}{n} = a(x, y) \) Formula of \( a(x, y) \) is not known for general r.v.

Exponential: \( a(x, y) = (\sqrt{x} + \sqrt{y})^2 \) [Rost]

Geometric: \( a(x, y) = \frac{q(x+y)+2\sqrt{qxy}}{1-q} \) [Johansson]

2. Limiting distribution

Exponential, Geometric [Johansson]

\[
\lim_{n \to \infty} \mathbb{P}\left( \frac{L([xn],[yn]) - a(x, y)n}{b(x, y)n^{1/3}} \leq s \right) = F_{TW}(s).
\]
Any general central limit theorem? Yes for thin models.

[Baik-Suidan] [Bodineau-Martin] [Suidan]

Suppose $E(X_{11}) = \mu$, $Var(X_{11}) = \sigma^2$ and $E|X_{11}|^4 < \infty$.

When $k = o(N^\alpha)$ and $\alpha < \frac{3}{14}$,

$$\lim_{N,k \to \infty} \mathbb{P}\left( \frac{L(N, k) - \mu(N + k - 1) - 2\sigma \sqrt{Nk}}{\sigma k^{-1/6} N^{1/2}} \leq s \right) = F(s).$$

If all the moments are finite, $\alpha < \frac{3}{7}$.
Proof

\( N \to \infty: \) each level \( \approx \) Brownian motion

\( k \) fixed, \( N \to \infty \) [Glynn + Whitt 1991]

\[
\frac{L(N, k) - \mu N}{\sigma \sqrt{N}} \Rightarrow \tilde{D}_k(1)
\]

\( \tilde{D}_k(1) := \sup_{0=t_0<t_1<\ldots<t_{N-1}<t_N:=1} \sum_{j=1}^{k} \left( B_j(t_j) - B_j(t_{j-1}) \right) \)

What is this functional?

‘Solvable’ case (exponential) [Baryshnikov 2001], [Gravner + Tracy + Widom]: \( \tilde{D}_k(1) = \) largest eigenvalue of \( k \times k \) random Hermitian matrix

Random matrix theory: \( k \to \infty \)

\( (\tilde{D}_k(1) - 2\sqrt{k})k^{1/6} \Rightarrow \chi \)

“\( k \to \infty, N \to \infty \)” \( \equiv \) “\( N, k \to \infty \)”?

Need to couple BM’s: Skorohod embedding, KMT (Komlós-Major-Tusnády) approximation