Boltzmann and Jeans’ Equations in Spherical Coordinates

Last time, we derived the collisionless Boltzmann equation, which was a kind of six-dimensional equation of continuity (though some terms were zero). We then took velocity moments, multiplying by powers of $v$ and then integrating over velocity space.

- The $v^0$ moment gave us a 3-dimensional equation of continuity.
- The $v_x$, $v_y$, and $v_z$ moments gave us 3 equations which are the collisionless analogues of hydrostatic equilibrium.
- We applied Jeans to the special case of plane parallel symmetry and found that the mass in the disk of the Milky Way isn’t sufficient to account for the circular velocity at the sun’s position.

1. Of course, galaxies have other symmetries besides plane parallel. It’s therefore useful to have the Boltzmann and Jean’s equations in other coordinate systems. We’ll do the spherical case, and let you ponder how you’d do the cylindrical version.

2. Spherical coordinates have a radius and two angles: $\theta$ and $\phi$. There are three associated velocities:
   - $v_r = \dot{r}$
   - $v_\theta = r\dot{\theta}$
   - $v_\phi = r\sin\theta\dot{\phi}$

Boltzmann:

$$0 = \frac{\partial f}{\partial t} + \vec{v} \cdot \vec{\nabla} f + \vec{\dot{v}} \cdot \vec{\nabla} v f$$

Hence, we’re going to need $\dot{r}$, $\dot{\theta}$, and $\dot{\phi}$.

3. Two ways to do this:
   (a) Write down the Lagrangian
   $$L = \frac{1}{2}mv^2 - \Phi$$
   and take
   $$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$$
   But, some of you haven’t had analytical classical mechanics!
   (b) We’ll assume spherical symmetry. We have 3 conserved quantities ($E$, $L^2$, and $L_z$) which, when differentiated, give 3 equations of motion.

4. $$E = \frac{1}{2} \left( \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 \right) + \Phi = \frac{1}{2} r^2 + \frac{L^2}{r^2} + \Phi$$

$$L^2 = r^4\dot{\theta}^2 + r^4\sin^2\theta\dot{\phi}^2 = r^4\dot{\theta}^2 + L_z^2$$

$$L_z = r^2\sin\theta\dot{\phi}$$

$$L = \vec{r} \times \vec{p} = \vec{r} \times \left( v_r\dot{r} + v_\theta\dot{\theta} + v_\phi\dot{\phi} \right)$$
= \underbrace{r^2 \dot{\phi}}_{\perp \text{to } z} + \underbrace{r^2 \sin \theta \ddot{\phi}}_{\text{projects as } -\sin \theta}

5. Now, each of our conserved quantities is expressed in terms of \( \dot{r}, \dot{\theta}, \dot{\phi} \) only.

- Solve for \( v_r, v_\theta, v_\phi \) and differentiate to get \( \dot{v}_r, \dot{v}_\theta, \dot{v}_\phi \).

\[
L_z = r \sin \theta v_\phi
\]
\[
\dot{L}_z = 0 = \dot{r} \sin \theta v_\phi + r \cos \theta \dot{\theta} v_\phi + r \sin \theta \ddot{\phi}
\]
\[
v_\phi = -\frac{v_r v_\phi}{r} - \cot \theta \frac{\theta v_\phi}{r} \left\{ -\frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \phi} \right\}
\]

- Likewise,

\[
\dot{v}_\theta = -\frac{v_r v_\theta}{r} + \cot \theta \frac{v_\phi^2}{r} \left\{ -\frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \phi} \right\}
\]

\[
\dot{v}_r = \frac{1}{r} (v_\theta^2 + v_\phi^2) - \frac{\partial \Phi}{\partial r}
\]

Chain rule: \( \frac{\partial \Phi}{\partial r} \).

The bracketed terms come from proper Lagrangian treatment.

Substitute these into

\[
\frac{\partial f}{\partial t} + v_r \frac{\partial f}{\partial r} + v_\theta \frac{\partial f}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial f}{\partial \phi} + \dot{v}_r \frac{\partial f}{\partial v_r} + \dot{v}_\theta \frac{\partial f}{\partial v_\theta} + \dot{v}_\phi \frac{\partial f}{\partial v_\phi} = 0
\]

6. Jeans’ equations are now straightforward but messy.

(a) zeroth moment: \( \int d^3v \)

(b) 3 first moments: \( \int v_r d^3v, \) etc.

(c) Let me report, without proof, the radial Jeans equation:

\[
\bar{v} = \int f d^3v
\]
\[
\bar{v}_r = \frac{1}{\nu} \int v_r f d^3v
\]

Et cetera.

\[
\sigma_{rr}^2 = \frac{1}{\nu} \int v_r^2 f d^3v
\]

\[
-\nu \frac{\partial \Phi}{\partial r} = \nu \frac{\partial \bar{v}_r}{\partial t} + \nu \left[ \bar{v}_R \frac{\partial \bar{v}_R}{\partial r} + \frac{\bar{v}_\theta}{r} \frac{\partial \bar{v}_r}{\partial \theta} + \frac{\bar{v}_\phi}{r \sin \theta} \frac{\partial \bar{v}_r}{\partial \phi} \right]
\]
\[
+ \frac{\partial}{\partial r} \left( \nu \sigma_{rr}^2 \right) + \frac{1}{r} \frac{\partial}{\partial \theta} \left( \nu \sigma_{r\theta}^2 \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left( \nu \sigma_{r\phi}^2 \right)
\]
\[
+ \frac{\nu}{r} \left[ 2 \sigma_{rR}^2 + \sigma_{r\theta}^2 + \sigma_{r\phi}^2 + \bar{v}_r^2 \right] + \sigma_{\phi}^2 \cot \theta
\]

It’s just acceleration is gradient of gravitational potential plus a few extra terms. Remember \( \bar{v}_R \) isn’t a single particle– it’s an average over orbits going every which way. Hence, the extra terms.
7. Let’s restrict ourselves to systems which are steady state and spherically symmetric; then
   (a) $\frac{\partial}{\partial t} = 0$
   (b) $\bar{v}_r = 0$ (otherwise net inward motion)
   (c) $\bar{v}_\theta = \bar{v}_\phi = 0$ Otherwise, symmetry is broken.
   (d) $\sigma^2_{r\theta} = \sigma^2_{r\phi} = 0$ Otherwise, not spherically symmetric.
   (e) $\sigma^2_{\phi\phi} = \sigma^2_{\theta\theta} = \sigma^2_{t1}$ Spherical symmetry.

$$\frac{1}{\nu} \frac{\partial}{\partial r} (\nu \sigma^2_{rr}) + \frac{2(\sigma^2_{rr} - \sigma^2_{t1})}{r} = -\frac{\partial \Phi}{\partial r}$$

$\sigma^2_{rr} << \sigma^2_{t1}$ Nearly circular
$\sigma^2_{rr} >> \sigma^2_{t1}$ Nearly radial
$\sigma^2_{rr} = \sigma^2_{t1}$ Isotropic: our old friend hydrostatic equilibrium.

8. We have

$$\frac{\partial \Phi}{\partial r} = -\frac{GM(r)}{r^2}$$

so we can solve for $M(r)$. Define

$$\beta(r) \equiv 1 - \frac{\sigma^2_{t1}}{\sigma^2_{rr}}$$

“anisotropy parameter”

$$M(\leq r) = -\frac{\sigma^2_{rr}}{G} \left[ \frac{d\ln \nu}{d\ln r} + \frac{d\ln \sigma^2_{rr}}{d\ln r} + 2\beta(r) \right]$$

Typically, $\frac{d\ln \nu}{d\ln r} = -3$, $\frac{d\ln \sigma^2_{rr}}{d\ln r} \approx 0$, and $2\beta(r) = 0$ (if isotropic).

9. Depends upon second moment. Assume Gaussian Doppler broadening, and fit Gaussians. $\sigma^2_{los}$. Hence, the masses of elliptical galaxies are more uncertain than the masses of spiral galaxies.
10. Idealized model: isothermal sphere (analog of the Lane-Emden equation): \( \sigma^2_{rr} = \sigma^2_{tt} = \sigma^2 \)

\[
\frac{\sigma^2}{\rho} \frac{d\rho}{dr} = -\frac{\partial \Phi}{\partial r} = \frac{GM(\leq r)}{r^2}
\]

Multiply by \( r^2 \) and differentiate.

\[
\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \rho \right) = -\frac{4\pi G\rho}{\sigma^2}
\]

Let \( \rho = \lambda e^\psi \)

\[
\left[ \frac{\sigma^2}{4\pi G\lambda} \right] \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \psi \right) = -e^\psi
\]

Let \( \left[ \frac{\sigma^2}{4\pi G\lambda} \right] = \alpha^2 \), and let \( \xi = \frac{r}{\alpha} \)

\[
\frac{1}{\xi} \frac{d}{d\xi} \xi^2 \frac{d}{d\xi} \psi = -e^{+\psi}
\]

Boundary conditions:
\( \psi(0) = 1 \) and \( \psi'(0) = 0 \).

\( r_c \equiv 3\alpha \) core radius.

\[
v_c^2 = \frac{GM(r)}{r}
\]

There are stellar dynamical analogs of polytropes if \( \sigma^2_{rr} = \sigma^2_{tt} \) — very special case.
11. Can use \( M(r) = k \frac{\sigma_r^2}{r^2} \), where \( k = 3 \pm 1.5 \) depending on details.

<table>
<thead>
<tr>
<th>Elliptical Galaxies</th>
<th>Clusters of Galaxies</th>
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</thead>
<tbody>
<tr>
<td>( \sigma_r = 300,\text{km/s} )</td>
<td>( \sigma_R = 1000,\text{km/s} )</td>
</tr>
<tr>
<td>( r = 10,\text{kpc} )</td>
<td>( R = 1,\text{Mpc} )</td>
</tr>
</tbody>
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\[
\frac{M_{\text{cluster}}}{NM_{\text{galaxy}}} = \frac{\sigma_R^2}{N \sigma_r^2} \frac{R}{r} \approx 10
\]

More missing mass. This is within clusters! There are three kinds of missing mass/dark matter: within galaxies, within clusters, and within the universe.