Simplex method

Solving linear programs

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1.041/1.200/11.544 Transportation: Foundations and Methods
Outline

1. Simplex method
   a. Basic and non-basic variables
   b. Intuition
   c. Pivoting
2. Reduction to canonical form
3. Problem: trailer production

Sources and further readings

• AMP - Chapter 2 Solving Linear Programs | Companion slides of Applied Mathematical Programming by Bradley, Hax, and Magnanti (Addison-Wesley, 1977) prepared by José Fernando Oliveira Maria Antónia Carravilla
A systematic procedure for solving linear programs – the simplex method

- Proceeds by moving from one feasible solution to another, at each step improving the value of the objective function.
- Terminates after a finite number of such transitions.
- Two important characteristics of the simplex method:
  - The method is robust.
    - It solves any linear program;
    - It detects redundant constraints in the problem formulation;
    - It identifies instances when the objective value is unbounded over the feasible region; and
    - It solves problems with one or more optimal solutions.
  - The method is also self-initiating.
    - It uses itself either to generate an appropriate feasible solution, as required, to start the method, or to show that the problem has no feasible solution.
  - The simplex method provides much more than just optimal solutions.
    - It indicates how the optimal solution varies as a function of the problem data (cost coefficients, constraint coefficients, and righthand-side data).
    - Information intimately related to a linear program called the "dual" to the given problem: the simplex method automatically solves this dual problem along with the given problem.
Simplex Method
— A Preview
The canonical form

Maximize \( z = 0x_1 + 0x_2 - 3x_3 - x_4 + 20 \),
subject to:

\[
\begin{align*}
x_1 & - 3x_3 + 3x_4 = 6 \\
x_2 & - 8x_3 + 4x_4 = 4 \\
x_j & \geq 0 \quad (j = 1, 2, 3, 4)
\end{align*}
\]

Any linear programming problem can be transformed so that it is in canonical form!

1. All decision variables are constrained to be nonnegative.
2. All constraints, except for the nonnegativity of decision variables, are stated as equalities.
3. The righthand-side coefficients are all nonnegative.
4. **One decision variable is isolated in each constraint with a +1 coefficient** (\( x_1 \) in constraint (1) and \( x_2 \) in constraint (2)). The variable isolated in a given constraint does not appear in any other constraint and appears with a zero coefficient in the objective function.
Discussion

- Given any values for $x_3$ and $x_4$, the values of $x_1$ and $x_2$ are determined uniquely by the equalities.
  - In fact, setting $x_3 = x_4 = 0$ immediately gives a feasible solution with $x_1 = 6$ and $x_2 = 4$.
  - Solutions such as these will play a central role in the simplex method and are referred to as basic feasible solutions.

- In general, given a canonical form for any linear program, a basic feasible solution is given by setting the variable isolated in constraint $j$, called the $j$th basic-variable, equal to the righthand side of the $j$th constraint and by setting the remaining variables, called nonbasic, all to zero.

- Collectively the basic variables are termed a basis.

Maximize $z = 0x_1 + 0x_2 - 3x_3 - x_4 + 20$, subject to:

- $x_1 - 3x_3 + 3x_4 = 6$
- $x_2 - 8x_3 + 4x_4 = 4$
- $x_j \geq 0 \quad (j = 1, 2, 3, 4)$
Discussion

In the example above, the basic feasible solution
\( x_1 = 6, x_2 = 4, x_3 = 0, x_4 = 0 \) is optimal.
- For any other feasible solution, \( x_3 \) and \( x_4 \) must remain nonnegative.
- Since their coefficients in the objective function are negative, if either \( x_3 \) or \( x_4 \) is positive, \( z \) will be less than 20.
- Thus, the maximum value for \( z \) is obtained when \( x_3 = x_4 = 0 \).

Maximize
\[
z = 0x_1 + 0x_2 - 3x_3 - x_4 + 20,
\]
subject to:
\[
\begin{align*}
x_1 - 3x_3 + 3x_4 &= 6 \\
x_2 - 8x_3 + 4x_4 &= 4 \\
x_j &\geq 0 \quad (j = 1, 2, 3, 4)
\end{align*}
\]
Optimality Criterion

- Suppose that, in a maximization problem, every nonbasic variable has a nonpositive coefficient in the objective function of a canonical form.
- Then the basic feasible solution given by the canonical form maximizes the objective function over the feasible region.

Maximize $z = 0x_1 + 0x_2 - 3x_3 - x_4 + 20$, subject to:

- $x_1 - 3x_3 + 3x_4 = 6$
- $x_2 - 8x_3 + 4x_4 = 4$
- $x_j \geq 0 \quad (j = 1, 2, 3, 4)$
Unbounded Objective Value

Maximize $z = 0x_1 + 0x_2 + 3x_3 - x_4 + 20,$
subject to:

$$\begin{align*}
    x_1 - 3x_3 + 3x_4 &= 6 \quad (1) \\
    x_2 - 8x_3 + 4x_4 &= 4 \quad (2) \\
    x_j &\geq 0 \quad (j = 1, 2, 3, 4)
\end{align*}$$

- Since $x_3$ now has a positive coefficient in the objective function, it appears promising to increase the value of $x_3$ as much as possible.
- Let us maintain $x_4 = 0,$ increase $x_3$ to a value $t$ to be determined, and update $x_1$ and $x_2$ to preserve feasibility.
Discussion

Maximize $z = 0x_1 + 0x_2 + 3x_3 - x_4 + 20$, subject to:

$$
\begin{align*}
    x_1 - 3x_3 + 3x_4 &= 6 \\
    x_2 - 8x_3 + 4x_4 &= 4 \\
    x_j &\geq 0 \quad (j = 1, 2, 3, 4)
\end{align*}
$$

- No matter how large $t$ becomes, $x_1$ and $x_2$ remain nonnegative. In fact, as $t$ approaches $+\infty$, $z$ approaches $+\infty$.
- In this case, the objective function is unbounded over the feasible region.
Unboundedness Criterion

Suppose that, in a maximization problem, some nonbasic variable has a positive coefficient in the objective function of a canonical form.

If that variable has negative or zero coefficients in all constraints, then the objective function is unbounded from above over the feasible region.

Maximize \( z = 0x_1 + 0x_2 + 3x_3 - x_4 + 20 \), subject to:

\[
\begin{align*}
x_1 - 3x_3 + 3x_4 &= 6 \\
x_2 - 8x_3 + 4x_4 &= 4 \\
x_j &\geq 0 \\ (j &= 1, 2, 3, 4)
\end{align*}
\]
Improving a Nonoptimal Solution

Maximize $z = 0x_1 + 0x_2 - 3x_3 + x_4 + 20$
subject to:

\[
\begin{align*}
  x_1 & - 3x_3 + 3x_4 = 6 \\
  x_2 & - 8x_3 + 4x_4 = 4 \\
  x_j & \geq 0 \quad (j = 1, 2, 3, 4)
\end{align*}
\]

- As $x_4$ increases, $z$ increases.
- Maintaining $x_3 = 0$, let us increase $x_4$ to a value $t$, and update $x_1$ and $x_2$ to preserve feasibility.
Discussion

Maximize $z = 0x_1 + 0x_2 - 3x_3 + x_4 + 20$, subject to:

\[
\begin{align*}
  x_1 - 3x_3 + 3x_4 &= 6 \\
  x_2 - 8x_3 + 4x_4 &= 4 \\
  x_j &\geq 0 \quad (j = 1, 2, 3, 4)
\end{align*}
\]

- If $x_1$ and $x_2$ are to remain nonnegative, we require:
  \[
  6 - 3t \geq 0, \quad \text{that is, } t \leq \frac{6}{3} = 2
  \]
  and
  \[
  4 - 4t \geq 0, \quad \text{that is, } t \leq \frac{4}{4} = 1
  \]
- Therefore, the largest value for $t$ that maintains a feasible solution is $t = 1$.
- When $t = 1$, the new solution becomes $x_1 = 3, x_2 = 0, x_3 = 0, x_4 = 1$, which has an associated value of $z = 21$ in the objective function.
Discussion

- Note that, in the new solution, $x_4$ has a positive value and $x_2$ has become zero.
- Since nonbasic variables were previously given zero values before, it appears that $x_4$ has replaced $x_2$ as a basic variable.
- In fact, it is fairly simple to manipulate Eqs. (1) and (2) algebraically to produce a new canonical form, where $x_1$ and $x_4$ become the basic variables.

Maximize $z = 0x_1 + 0x_2 - 3x_3 + x_4 + 20$, subject to:

\[
egin{align*}
    x_1 - 3x_3 + 3x_4 &= 6 \\
    x_2 - 8x_3 + 4x_4 &= 4 \\
    x_j &\geq 0 \quad (j = 1, 2, 3, 4)
\end{align*}
\]
Discussion

- If $x_4$ is to become a basic variable, it should appear with coefficient +1 in Eq. (2), and with zero coefficients in Eq. (1) and in the objective function.
- To obtain a +1 coefficient in Eq. (2), we divide that equation by 4.

\[(1) \quad x_1 - 3x_3 + 3x_4 = 6 \quad x_1 \quad - 3x_3 + 3x_4 = 6\]
\[(2) \quad x_2 - 8x_3 + 4x_4 = 4 \quad \frac{1}{4}x_2 - 2x_3 + x_4 = 1\]
Discussion

To eliminate $x_4$ from the first constraint, we may multiply Eq. (2’) by 3 and subtract it from constraint (1).

\[
\begin{align*}
(1) & \quad x_1 - 3x_3 + 3x_4 = 6 \\
(2') & \quad \frac{1}{4}x_2 - 2x_3 + x_4 = 1
\end{align*}
\]

We may rearrange the objective function and write it as:

\[
(-z) - 3x_3 + x_4 = -20
\]

and use the same technique to eliminate $x_4$; that is, multiply (2’) by 1 and subtract it from the above:

\[
(-z) - \frac{1}{4}x_2 - x_3 = -21
\]
The new global system becomes

Maximize $z = 0x_1 - \frac{1}{4}x_2 - x_3 + 0x_4 + 21,$

subject to:

$x_1 - \frac{3}{4}x_2 - 3x_3 = 3$

$\frac{1}{4}x_2 - 2x_3 + x_4 = 1$

$x_j \geq 0 \quad (j = 1, 2, 3, 4)$

- Now the problem is in canonical form with $x_1$ and $x_4$ as basic variables, and $z$ has increased from 20 to 21.
- Consequently, we are in a position to reapply the arguments of this section, beginning with this improved solution.
- However, in this case, the new canonical form satisfies the optimality criterion since all nonbasic variables have nonpositive coefficients in the objective function, and thus the basic feasible solution $x_1 = 3, x_2 = 0, x_3 = 0, x_4 = 1,$ is optimal.
Improvement Criterion

- Suppose that, in a maximization problem, some nonbasic variable has a positive coefficient in the objective function of a canonical form.
- If that variable has a positive coefficient in some constraint, then a new basic feasible solution may be obtained by pivoting.
Discussion

- Recall that we chose the constraint to pivot in (and consequently the variable to drop from the basis) by determining which basic variable first goes to zero as we increase the nonbasic variable $x_4$.
- The constraint is selected by taking the ratio of the righthand-side coefficients to the coefficients of $x_4$ in the constraints, i.e., by performing the ratio test:

$$\min \left\{ \frac{6}{3}, \frac{4}{4} \right\}$$
Discussion

- Note, however, that if the coefficient of $x_4$ in the second constraint were $-4$ instead of $+4$, the values for $x_1$ and $x_2$ would be given by:

  $x_1 = 6 - 3t$
  $x_2 = 4 + 4t$

  so that as $x_4 = t$ increases from 0, $x_2$ never becomes zero. In this case, we would increase $x_4$ to $t = \frac{6}{3} = 2$.

- This observation applies in general for any number of constraints, so that we need never compute ratios for nonpositive coefficients of the variable that is coming into the basis.
Ratio and Pivoting Criterion

- When improving a given canonical form by introducing variable $x_s$ into the basis, pivot in a constraint that gives the minimum ratio of righthand-side coefficient to corresponding $x_s$ coefficient.
- Compute these ratios only for constraints that have a positive coefficient for $x_s$. 
Reduction to Canonical Form
Reduction to Canonical Form

- To this point we have been solving linear programs posed in canonical form with
  1. nonnegative variables,
  2. equality constraints,
  3. nonnegative righthand-side coefficients, and
  4. one basic variable isolated in each constraint.

- We will now show how to transform any linear program to this canonical form.
Inequality constraints

\[ 40x_1 + 10x_2 + 6x_3 \leq 55.0, \]
\[ 40x_1 + 10x_2 + 6x_3 \geq 32.5 \]

- Introduce two new nonnegative variables:
- \( x_5 \) measures the amount that the consumption of resource falls short of the maximum available, and is called a slack variable;
- \( x_6 \) is the amount of product in excess of the minimum requirement and is called a surplus variable.

\[ 40x_1 + 10x_2 + 6x_3 + x_5 = 55.0, \]
\[ 40x_1 + 10x_2 + 6x_3 - x_6 = 32.5 \]
Simplex Method
— A Simple Example
A simple example

- The owner of a shop producing automobile trailers wishes to determine the best mix for his three products:
  - flat-bed trailers
  - economy trailers
  - luxury trailers.

- His shop is limited to working 24 days/month on metalworking and 60 days/month on woodworking for these products. The following table indicates production data for the trailers.

<table>
<thead>
<tr>
<th></th>
<th>Usage per unit of trailer</th>
<th>Resources availabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flat-bed</td>
<td>Economic</td>
<td>Luxury</td>
</tr>
<tr>
<td>Metalworking days</td>
<td>1/2</td>
<td>2</td>
</tr>
<tr>
<td>Woodworking days</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>Contribution ($ \times 100$)</td>
<td>6</td>
<td>14</td>
</tr>
</tbody>
</table>
LP Model

- Let the decision variables of the problem be:
  - $x_1 =$ Number of flat-bed trailers produced per month
  - $x_2 =$ Number of economy trailers produced per month
  - $x_3 =$ Number of luxury trailers produced per month

Maximize $z = 6x_1 + 14x_2 + 13x_3$, 
subject to:

$$\frac{1}{2}x_1 + 2x_2 + x_3 \leq 24$$

$$x_1 + 2x_2 + 4x_3 \leq 60$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0$$
Canonical form

Maximize \( z = 6x_1 + 14x_2 + 13x_3 \),
subject to:

\[
\begin{align*}
\frac{1}{2}x_1 + 2x_2 + x_3 + x_4 &= 24 \\
x_1 + 2x_2 + 4x_3 + x_5 &= 60 \\
x_j &\geq 0 \quad (j = 1, 2, 3, 4, 5)
\end{align*}
\]

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Current values</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_4 )</td>
<td>24</td>
<td>( \frac{1}{2} )</td>
<td>( \text{(2)} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( x_5 )</td>
<td>60</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( -z )</td>
<td>0</td>
<td>+6</td>
<td>+14</td>
<td>+13</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\((-z) + 6x_1 + 14x_2 + 13x_3 = 0.\)
# Iterations

## Tableau 1

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Current values</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_4$</td>
<td>24</td>
<td>$\frac{1}{2}$</td>
<td>(2)</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$x_5$</td>
<td>60</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>($-z$)</td>
<td>0</td>
<td>+6</td>
<td>+14</td>
<td>+13</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Equation identification and transformations:
- $\frac{1}{x_4}$
- $\frac{2}{x_5}$
- $\frac{3}{3}$

Ratio test:
- $24/2$
- $60/2$

## Tableau 2

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Current values</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>12</td>
<td>$\frac{1}{4}$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>$x_5$</td>
<td>36</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
</tr>
<tr>
<td>($-z$)</td>
<td>-168</td>
<td>+3</td>
<td>+6</td>
<td>-7</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Equation identification and transformations:
- $4 = \frac{1}{2}$
- $5 = 2 - \frac{4}{3}$
- $6 = 3 - \frac{4}{4}$

Ratio test:
- $12/(1/2)$
- $36/3$
# Iterations

## Tableau 3

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Current values</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>6</td>
<td>$\frac{1}{6}$</td>
<td>1</td>
<td>1</td>
<td>$\frac{2}{3}$</td>
<td>$-\frac{1}{6}$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>12</td>
<td>$\frac{1}{2}$</td>
<td></td>
<td></td>
<td>$-\frac{1}{3}$</td>
<td>$\frac{1}{3}$</td>
</tr>
<tr>
<td>($-z$)</td>
<td>-240</td>
<td></td>
<td></td>
<td>-5</td>
<td>-2</td>
<td></td>
</tr>
</tbody>
</table>

Equation identification and transformations:

- $7 = 4 - \frac{1}{2} \times 8$
- $8 = \frac{1}{3} \times 5$
- $9 = \frac{1}{6} \times 6 - 6 \times 8$

Ratio test:
- 6/(1/6)
- 12/(1/6)

## Tableau 4

<table>
<thead>
<tr>
<th>Basic variables</th>
<th>Current values</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>36</td>
<td>1</td>
<td>6</td>
<td>4</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>$x_3$</td>
<td>6</td>
<td></td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>$\frac{1}{2}$</td>
</tr>
<tr>
<td>($-z$)</td>
<td>-294</td>
<td></td>
<td>-9</td>
<td>-11</td>
<td>-1</td>
<td>$-\frac{1}{2}$</td>
</tr>
</tbody>
</table>

Equation identification and transformations:

- $10 = 6 \times 7$
- $11 = 8 - \frac{1}{6} \times 10$
- $12 = 9 - \frac{3}{2} \times 10$
Minimization problems

- Enters the basis the nonbasic variable that has a **negative** coefficient in the objective function of a canonical form.
- The solution is optimal when every nonbasic variable has a **nonnegative** coefficient in the objective function of a canonical form.
Formal Procedure

Simplex Algorithm (Maximization Form)

0. The problem is initially in canonical form and all \( \bar{b}_i \geq 0 \).

1. If \( \bar{c}_j \leq 0 \) for \( j = 1, 2, \ldots, n \), then stop; we are optimal. If we continue then there exists some \( \bar{c}_j > 0 \).

2. Choose the column to pivot in (i.e. the variable to introduce into the basis) by:
   \[
   \bar{c}_s = \max \{ \bar{c}_j | \bar{c}_j > 0 \}
   \]
   If \( \bar{a}_{is} \leq 0 \) for \( i = 1, 2, \ldots, m \), then stop; the primal problem is unbounded. If we continue, then \( \bar{a}_{is} > 0 \) for some \( i = 1, 2, \ldots, m \).

3. Choose row \( r \) to pivot in (i.e. the variable to drop from the basis) by the ratio test:
   \[
   \frac{\bar{b}_r}{\bar{a}_{rs}} \min \left\{ \frac{\bar{b}_i}{\bar{a}_{is}} \left| \bar{a}_{is} > 0 \right. \right\}
   \]

4. Replace the basic variable in row \( r \) with variable \( s \) and reestablish the canonical form (i.e. pivot on the coefficient \( \bar{a}_{rs} \)).

5. Go to step (1).
STEP (4) Pivoting

\[
\begin{array}{cccc|cccc|c}
\hline
x_1 & \cdots & x_r & \cdots & x_{m+1} & \cdots & x_s & \cdots & x_n \\
\hline
1 & \cdots & \cdots & \bar{a}_{1,m+1} & \cdots & \bar{a}_{1s} & \cdots & \bar{a}_{1n} & \bar{b}_1 \\
\vdots & \cdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
1 & \cdots & \cdots & \bar{a}_{r,m+1} & \cdots & \bar{a}_{rs} & \cdots & \bar{a}_{rn} & \bar{b}_r \\
\vdots & \cdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
1 & \cdots & \cdots & \bar{a}_{m,m+1} & \cdots & \bar{a}_{ms} & \cdots & \bar{a}_{mn} & \bar{b}_m \\
\bar{c}_{m+1} & \cdots & \bar{c}_s & \cdots & \bar{c}_n & -\bar{z}_0 \\
\hline
\end{array}
\]

\begin{align*}
\downarrow \text{Normalization} \\
1 & \cdots & \cdots & \left( \frac{1}{a_{rs}} \right) & \cdots & \cdots & \left( \frac{1}{a_{rn}} \right) & \cdots & \cdots \\
\vdots & \cdots & \cdots & \vdots & \cdots & \vdots & \cdots & \cdots & \cdots \\
1 & \cdots & \cdots & \left( \frac{a_{r,m+1}}{a_{rs}} \right) & \cdots & 1 & \cdots & \left( \frac{a_{rn}}{a_{rs}} \right) & \left( \frac{b_r}{a_{rs}} \right) \\
\vdots & \cdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots & \vdots \\
1 & \cdots & \cdots & \left( \frac{a_{m,m+1}}{a_{ms}} \right) & \cdots & \cdots & \left( \frac{a_{mn}}{a_{ms}} \right) & \bar{b}_m \\
\bar{c}_{m+1} & \cdots & \bar{c}_s & \cdots & \bar{c}_n & -\bar{z}_0 \\
\end{align*}
STEP (4) Pivoting

\[
\begin{array}{c|c|c}
1 & \bar{a}_{1, m+1} & \ldots & \bar{a}_{1s} & \ldots & \bar{a}_{1n} & \bar{b}_1 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\left(\frac{1}{\bar{a}_{rs}}\right) & \left(\frac{\bar{a}_{r, m+1}}{\bar{a}_{rs}}\right) & \ldots & 1 & \ldots & \left(\frac{\bar{a}_{rn}}{\bar{a}_{rs}}\right) & \left(\frac{\bar{b}_r}{\bar{a}_{rs}}\right) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\bar{a}_{m, m+1} & \ldots & \bar{a}_{ms} & \ldots & \bar{a}_{mn} & \bar{b}_m \\
\bar{c}_{m+1} & \ldots & \bar{c}_s & \ldots & \bar{c}_n & -\bar{c}_0 \\
\end{array}
\]

↓ Elimination of \(x_s\)

\[
\begin{array}{c|c|c}
1 & -\left(\frac{\bar{a}_{1s}}{\bar{a}_{rs}}\right) & \bar{a}_{1, m+1} - \bar{a}_{1s} \left(\frac{\bar{a}_{r, m+1}}{\bar{a}_{rs}}\right) & \ldots & 0 & \ldots & \bar{a}_{1n} - \bar{a}_{1s} \left(\frac{\bar{a}_{rn}}{\bar{a}_{rs}}\right) & \bar{b}_1 - \bar{a}_{1s} \left(\frac{\bar{b}_r}{\bar{a}_{rs}}\right) \\
\vdots & \vdots & \bar{a}_{m, m+1} - \bar{a}_{ms} \left(\frac{\bar{a}_{r, m+1}}{\bar{a}_{rs}}\right) & \ldots & 0 & \ldots & \bar{a}_{mn} - \bar{a}_{ms} \left(\frac{\bar{a}_{rn}}{\bar{a}_{rs}}\right) & \bar{b}_m - \bar{a}_{ms} \left(\frac{\bar{b}_r}{\bar{a}_{rs}}\right) \\
\left(\frac{1}{\bar{a}_{rs}}\right) & \left(\frac{\bar{a}_{r, m+1}}{\bar{a}_{rs}}\right) & \ldots & 1 & \ldots & \left(\frac{\bar{a}_{rn}}{\bar{a}_{rs}}\right) & \left(\frac{\bar{b}_r}{\bar{a}_{rs}}\right) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
-\left(\frac{\bar{c}_s}{\bar{a}_{rs}}\right) & 1 & \bar{c}_{m+1} - \bar{c}_s \left(\frac{\bar{a}_{r, m+1}}{\bar{a}_{rs}}\right) & \ldots & 0 & \ldots & \bar{c}_n - \bar{c}_s \left(\frac{\bar{a}_{rn}}{\bar{a}_{rs}}\right) & -\bar{c}_0 - \bar{c}_s \left(\frac{\bar{b}_r}{\bar{a}_{rs}}\right) \\
\end{array}
\]