Linear programming I

Modeling mathematical programs

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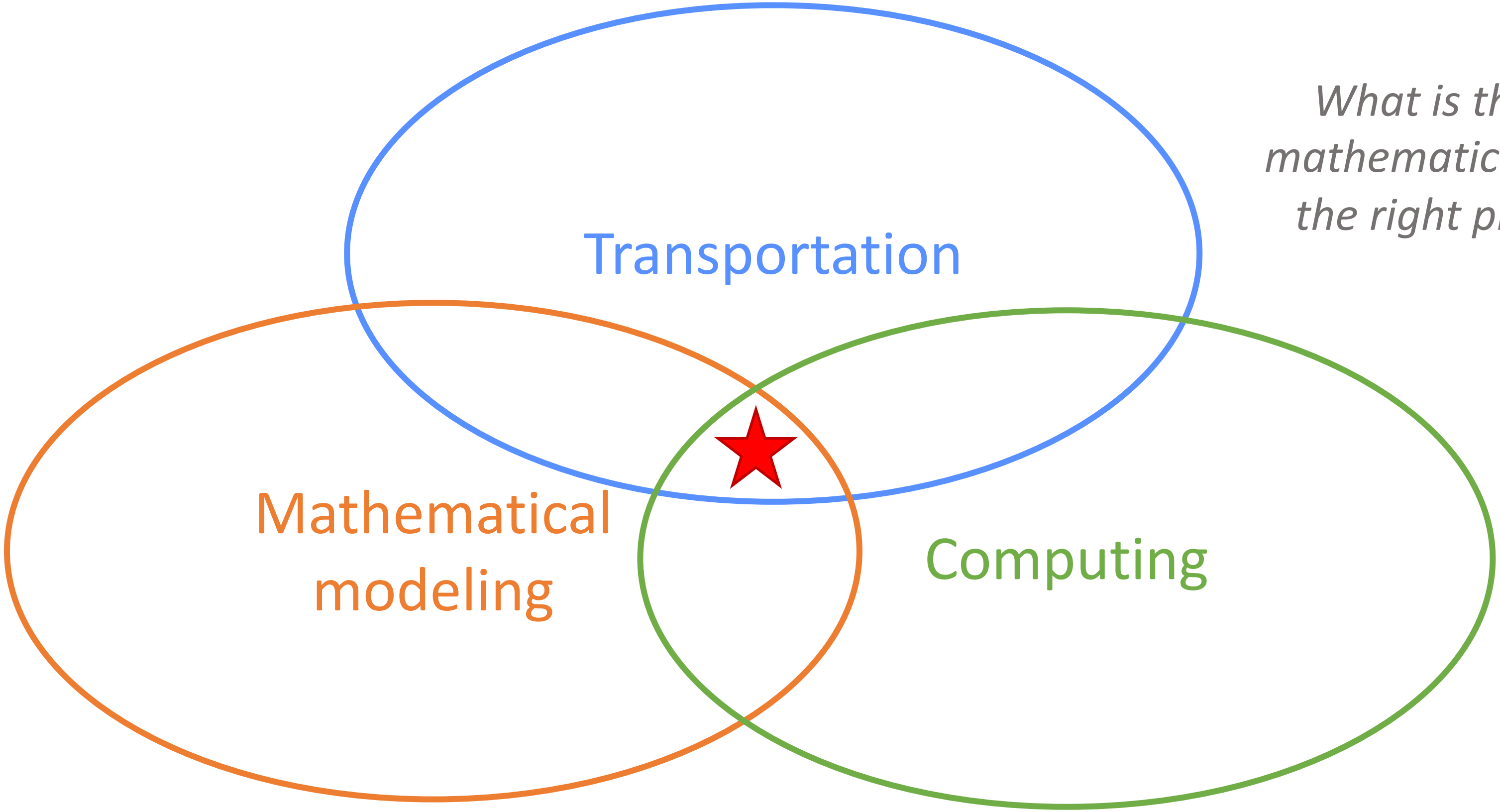
1.041/1.200 Transportation: Foundations and Methods

Readings

- Bradley, Stephen P., Arnoldo C. Hax, and Thomas L. Magnanti. **Applied mathematical programming**. Addison-Wesley (1977). Chapter 1 Mathematical Programming: an overview [URL]

Transportation: Foundations & Methods



What is the right mathematical tool for the right problem?

Big picture overview of the course

Model transportation systems

Optimize transportation systems

Unit 1

Unit 2

Unit 3

Unit 4

Deterministic

Stochastic

Multi-stage

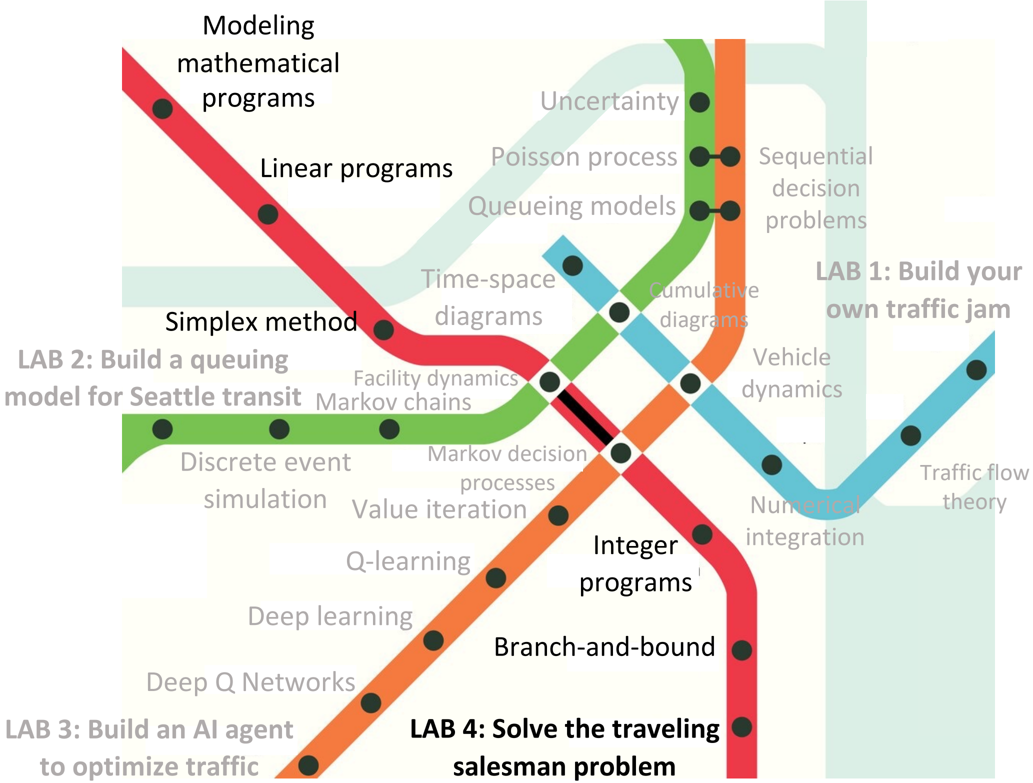
Single-stage

Unit 4: Optimizing transportation resources

Unit 4

Optimizing

Single-stage



Outline

1. Linear programming
2. Linearizing non-linear problems
3. Modeling caveats and standard form

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1. **Linear programming**
   a. Transit ridership problem
2. Linearizing non-linear problems
3. Modeling caveats and standard form

Mathematical programming (Bradley et al., 1977, Chapter 1)

Mathematical programming, and especially linear programming, is one of the best developed and most used branches of management science.

It concerns the **optimum allocation of limited resources** among competing activities, **under a set of constraints** imposed by the nature of the problem being studied.

Main components of an optimization problem

1. Objective function
   - summarizes the objective of the problem (max, min)
2. Constraints
   - limitations placed on the problem; control allowable solutions
   - problem statement: ‘given…’, ‘must ensure…’, ‘subject to’
   - equations or inequalities or variable value types
3. Decision variables
   - quantities, decisions to be determined
   - multiple types (real numbers, non-negative, integer, binary)

An optimization problem with linear objective function AND linear constraints is called a **linear program** / a linear optimization problem

Linear Programming model

- There are n quantifiable decisions to be made. Choose their values such as to min/max an objective. $(x_1, x_2, \dots, x_n)$ decision variables

$$
\begin{aligned}
&\min c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
&s.t. \\
&a_{11}x_1 + a_{12}x_2 + \ \dots + a_{1n}x_n = b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ \dots + a_{2n}x_n = b_2 \\
&\dots \\
&a_{m1}x_1 + a_{m2}x_2 + \ \dots a_{mn}x_n = b_m \\
&x_i \geq 0
\end{aligned}
$$

- x_j: level of activity j (for $j = 1, 2, \dots, n$)
- c_j: increase in objective function that would result from each unit increase in the level of the activity j
- b_i: amount of resource i that is available for allocation to activities (m resources, $i = 1, 2, \dots, m$)
- a_{ij}: amount of resource i consumed by each unit of activity j
- Model parameters: c_j, b_i, a_{ij}. They are input constants for the model.

Linear program

$$\max_{x} c^T x$$
$$s.t.\ Ax = b$$
$$x \geq 0$$

where $x \in \mathbb{R}^n, c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, and $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$

General mathematical programs

A general (possibly nonlinear) mathematical program is of the form:

$$
\begin{aligned}
\max_{x}\ & f_0(x) \\
s.t.\ & f_1(x) \le b_1 \\
& f_2(x) \le b_2 \\
& \vdots \\
& f_m(x) \le b_m \\
& x \in X
\end{aligned}
$$

where X denotes a (possibly infinite) set of valid variable scopes.

Mixed integer linear program

$$\begin{aligned} \max_{x,y}\, c^T x + d^T y & \\ s.t.\ Ax &= b \\ Cy &= e \\ x, y &\geq 0 \\ y &\in \mathbb{Z}_+^{n_1} \end{aligned}$$

where A, b, c are as before,

and d $= \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n_1} \end{bmatrix}$, $C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n_1} \\ c_{21} & c_{22} & \cdots & c_{2n_1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{m_1 n_1} \end{bmatrix}$, and $e = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_{m_1} \end{bmatrix}$

Note that the constraints and objective remain linear.

Example: transit ridership

- A transit agency is performing a review of its services. It has decided to measure its overall effectiveness in terms of the total number of riders it serves. The agency operates a number of modes of transport. The table shows the average number of riders generated by each trip (by mode) and the cost of each trip (by mode).

Mode	Heavy rail	Light rail	BRT	Bus
Avg. ridership per trip (r_i)	400	125	60	40
Avg. cost ($) per trip (c_i)	200	80	40	30

- Give a formulation of the problem to <u>maximize the total average number of riders</u> the agency services <u>given a fixed daily budget of $5,000</u>.

Outline

1. Mathematical programming
2. Linear programming
3. **Linearizing non-linear problems**
   a. Ride-hailing problem
4. Modeling caveats and standard form

Non-linearity

- In a Linear Program (LP),
  - objective function AND constraints MUST BE linear
  - variable type must be continuous
  - e.g. real numbers, non-negatives are OK
  - integers, binary NOT OK
- $\max\{x_1, x_2, \ldots\}$, $x_i y_i$, $|x_i|$, etc. $\Rightarrow$ non-linear (if x_i and y_i are both variables)
  - sometimes there is a way to convert these types of constraints into linear constraints by adding some decision variables

Non-linear programming

$$\text{Maximize } z = 60x_1 - 5x_1^2 + 80x_2 - 4x_2^2$$

subject to:

$$6x_1 + 5x_2 \leq 60$$
$$10x_1 + 20x_2 \leq 150$$
$$8 \geq |x_1| \geq 0$$
$$x_2 \geq 0$$

Dealing with absolute values

Goal: We want to formulate the following as a linear program

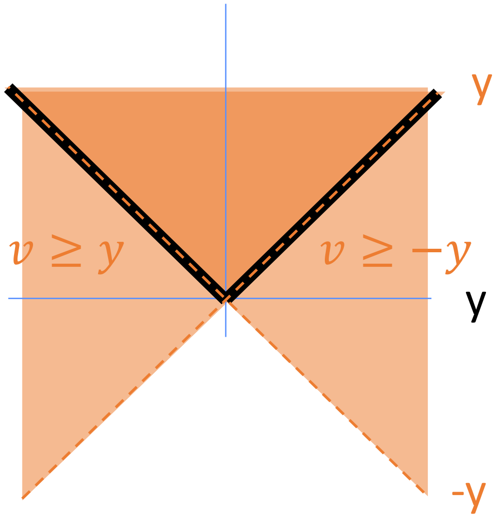
$$\min 5x + 2|y|$$
$$s.t.$$
$$x + y \geq 9$$

- Option 1

$$|y| = \max\{y, -y\}$$

- Option 2
  - Add a new variable v
  - Replace by $v \geq y$ and $v \geq -y$
- Formulation 1:

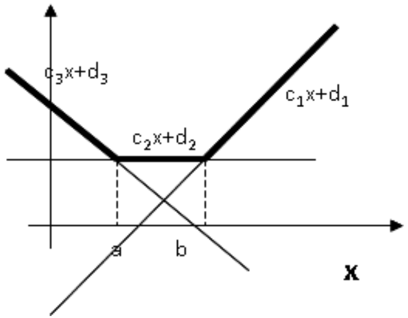
$$\min 5x + 2v$$
$$s.t.$$
$$v \geq y$$
$$v \geq -y$$
$$x + y \geq 9$$



Minimizing piecewise linear convex cost functions

<u>Example</u>

Costs defined as: $\begin{cases} c(x) = c_3 x + d_3, & \forall x \in [-\infty, a] \\ c(x) = c_2 x + d_2, & \forall x \in [a, b] \\ c(x) = c_1 x + d_1, & \forall x \in [b, +\infty] \end{cases}$



What to do?

Introduce a new variable t with objective min t such that:

$$t = \max\{c_3 x + d_3, c_2 x + d_2, c_1 x + d_1\}$$

In linear form: $\begin{cases} t \geq c_3 x + d_3 \\ t \geq c_2 x + d_2 \\ t \geq c_1 x + d_1 \end{cases}$

<u>Exercise</u>: Construct an argument for the validity of this formulation

Ride Hailing Problem

- A new startup ride hailing company DropMe, has a budget of \$150,000 to expand its taxi fleet.
- In order to increase their revenue, the firm is considering expanding its taxi fleet so that it can cover more suburban areas.
- Diminishing marginal returns: However, the larger the fleet, the less effective the reach of new passengers.
- DropMe is considering of adding two types of vehicles to its fleet (type A and type B) to combat the diminishing marginal returns by maintaining a certain "cool factor."
- Each addition of a type A vehicle costs \$1,000 and each addition of type B vehicle costs \$10,000.
- At most 30 more type A vehicles and at most 15 more type B vehicles can be added to DropMe fleet due to state law restrictions.
- The objective is to maximize the number of new passengers reached through this new fleet.
- Formulate this problem as a linear program.

	Nb. of additions	Nb. of new passengers reached
Type A	1 – 10	900
	11 – 20	600
	21 – 30	300
Type B	1 – 5	10,000
	6 – 10	5,000
	11 – 15	2,000

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Caveats on mathematical programming model

It's a model!

- It's an abstract idealization of the real problem
- Simplifying assumptions and approximations are often needed to obtain a tractable model (i.e. model that can be solved in a reasonable amount of time)
- Models provide insights rather than accurate quantitative forecasts

Caveats on mathematical programming model

"Good model"

- Predicts the relative effects of alternative decisions with sufficient accuracy to permit a sound decision
- There is high correlation between the model predictions and what would actually happen in the real world, as opposed to high accuracy in the model forecasts

Caveats on mathematical programming model

"Optimal solutions"

- Are optimal with regards to the underlying model, which is a simplification of reality
- There is no guarantee that this solution will prove to be the best possible solution when implemented, because there are just too many uncertainties associated to the real problem
- A well formulated and validated (tested) model should lead to solutions that are a good approximation to an ideal course of action for the real problem

$$
\begin{aligned}
&\min c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
&s.t. \\
&\quad a_{11}x_1 + a_{12}x_2 + \ \ldots + a_{1n}x_n = b_1 \\
&\quad a_{21}x_1 + a_{22}x_2 + \ \ldots + a_{2n}x_n = b_2 \\
&\quad \ldots \\
&\quad a_{m1}x_1 + a_{m2}x_2 + \ \ldots a_{mn}x_n = b_m \\
&\quad x_i \geq 0
\end{aligned}
$$

Assumptions of the Linear Programming model

Main assumptions:

- Proportionality
- Additivity
- Divisibility
- Certainty

Does a linear programming formulation provide a satisfactory representation of a problem?

Converting constraints to standard form

Standard form

$$\min c^T x$$
$$s.t.\ \ A\,x = b$$
$$x \geq 0$$

Non-standard form

- What about $a_1 x_1 + a_2 x_2 \geq b_1$?
- Replace with:
  - $x_3 := a_1 x_1 + a_2 x_2 - b_1$ (constraint)
  - Add constraint: $x_3 \geq 0$

- What about $x_1 \leq 0$?
- Replace with:
  - Define $x_2 := -x_1$ (constraint)
  - Add constraint $x_2 \geq 0$.

- Why is this important?
  - Solvers (software) often require LPs to be formulated in standard form.

References

1. AMP - Chapter 1 Mathematical Programming: an overview Companion slides of Applied Mathematical Programming by Bradley, Hax, and Magnanti (Addison-Wesley, 1977) prepared by José Fernando Oliveira Maria Antónia Carravilla
2. Lecture slides from Prof Carolina Osorio (MIT 1.041)
3. Introduction to Linear Optimization (Bertsimas and Tsitsiklis) available in library
4. Formulating an MP: An Overview (Nathaniel Grier) available through Piazza
5. Introduction to Operations Research (Hillier and Lierberman)

Additional exercises

Example: emergency response

- An emergency response center is to be located in a region with 4 major communities. The communities are located at
$$(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$$
  1. Choose the optimal location of emergency response center to minimize the **sum of distances** from each community
  2. Choose the optimal location of emergency response center to minimize the **maximum distance** from each community
- Formulate the above as linear programs.
- These are also known as **facility location problems**.
- Use the Manhattan distance
  - Consider 2 points:
$$\mathrm{p}_1 = (x_1, y_1), \mathrm{p}_2 = (x_2, y_2)$$
  - Manhattan distance metrics
$$\mathrm{d}(\mathrm{p}_1, \mathrm{p}_2) = |x_1 - x_2| + |y_1 - y_2|$$
  Makes sense in cases of grid-like road systems (like Manhattan)