Spring 2024

Linear Programming II

Geometric Solutions

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1.041/1.200 Transportation: Foundations and Methods

Readings

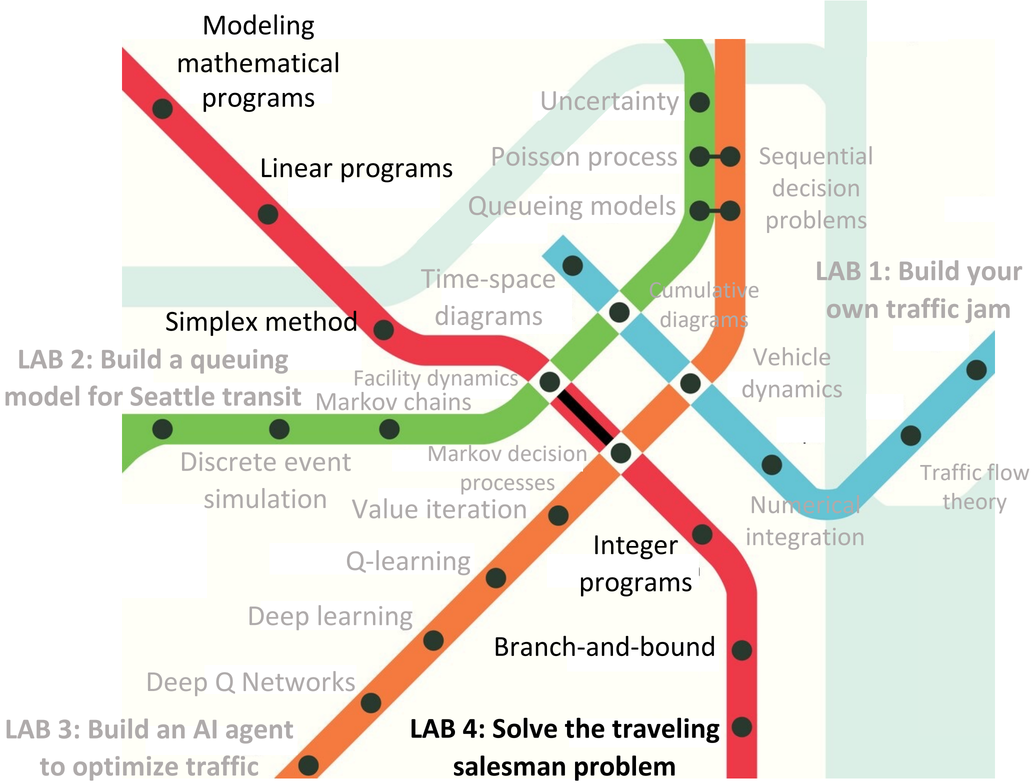
- Bradley, Stephen P., Arnoldo C. Hax, and Thomas L. Magnanti. **Applied mathematical programming**. Addison-Wesley (1977). Chapter 1 Mathematical Programming: an overview [URL]

Unit 4: Optimizing transportation resources

Unit 4

Optimizing

Single-stage



Outline

1. Geometric solutions for 2D problems

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1. **Geometric solutions for 2D problems**
   a. Delivery route planning problem
   b. Problem and solution sensitivity
   c. Problem feasibility

Example: Delivery Route Planning

Cambridge Deliveries, a company that provides heavy load delivery services, is planning on undertaking two new delivery routes for two new clients, **A** and **B**.

- Client B proposed a payment of $100 per delivery load, while Client A proposed $30 per delivery load. A delivery of Client A weighs 0.5T (tons) and 2T for Client B.
- Based on the type of vehicles that will be used for these deliveries, Cambridge Deliveries expects to use 2L of fuel per delivery for Client A, and 5L of fuel per delivery for Client B.
- Assume that Cambridge Deliveries has a long-term contract with the drivers, so driver cost can be assumed to be a fixed cost.
- The cost of fuel is $5/L.
- Cambridge Deliveries will have to pay a state tax of $12/T of any delivery as a compensation for wear and tear of roads due to heavy vehicles.
- According to Cambridge Deliveries' budget, they may consume up to 6000L of fuel and carry a maximum of 3000T of delivery loads.
- Client A has 2000 deliveries planned while Client B has 800.

Example: Delivery Route Planning

Problem:

1. Write the mathematical model of this LP to maximize the profit for Cambridge Deliveries.
2. Number of deliveries has to be integral. What is the relation of the optimal profit to #1?
3. The company is now thinking of taking another route per Client C. As per the agreement with Client C, it will bear the cost of fuel and thus Cambridge Deliveries will incur zero cost for fuel. Client C agreed to pay $500 per delivery and will offer 99 deliveries in total. However each delivery weighs 10T. What is the relation of the optimal profit to (1)?
4. A) # of deliveries for Client B can be at most 400.
   b) # of deliveries for Client B has to be at least 401.
   What is the relation of the optimal profits of #4A and #4B to #2?

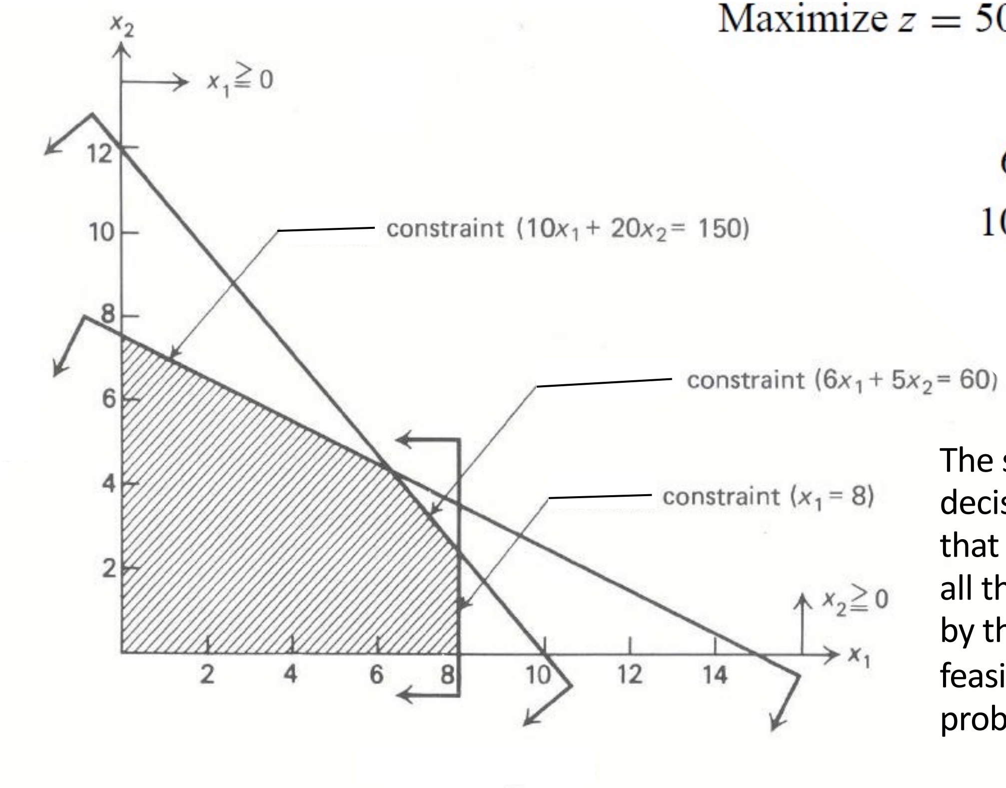
Example: Delivery Route Planning

- Budget: 6000L fuel, 3000T delivery load
- Cost: $5/L fuel, $12/T

1. Write the mathematical model of this LP to maximize the profit for Cambridge Deliveries.

	Client A	Client B
$ / delivery	$30	$100
Weight / delivery	0.5T	2T
Fuel / delivery	2L	5L

Graphical Representation of the Decision Space



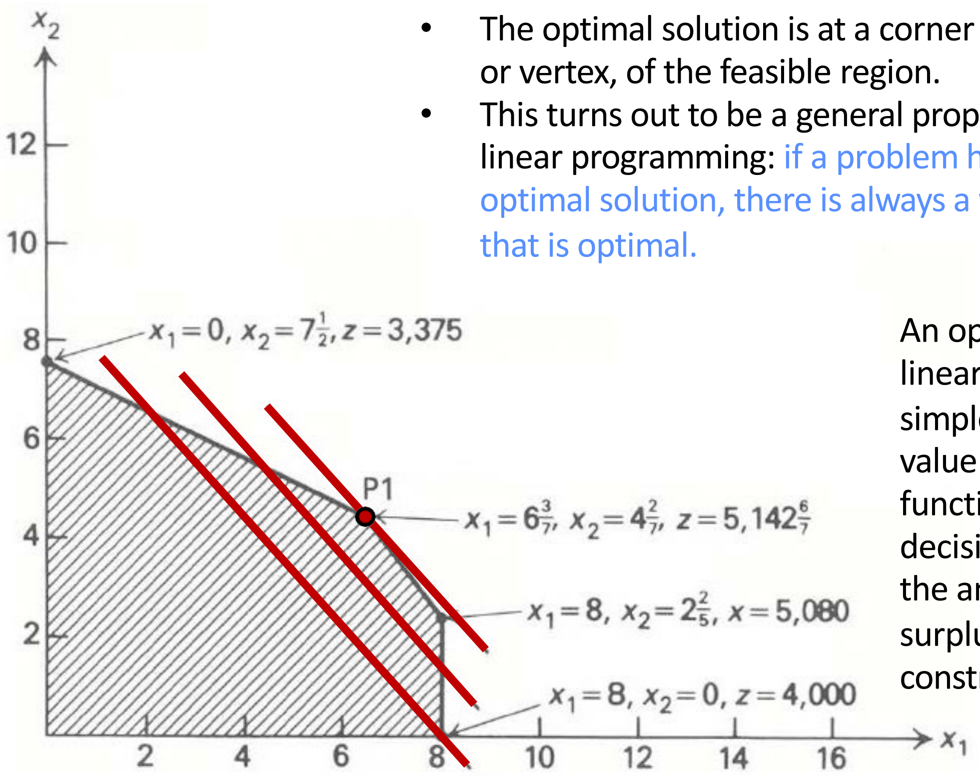
$$\text{Maximize } z = 500x_1 + 450x_2,$$

$$\begin{aligned} 6x_1 + 5x_2 &\leq 60 \\ 10x_1 + 20x_2 &\leq 150 \\ x_1 &\leq 8 \end{aligned}$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$

The set of values of the decision variables x_1 and x_2 that simultaneously satisfy all the constraints indicated by the shaded area are the feasible solutions to the problem.

Finding an optimal solution



- The optimal solution is at a corner point, or vertex, of the feasible region.
- This turns out to be a general property of linear programming: if a problem has an optimal solution, there is always a vertex that is optimal.

An optimal solution of a linear program in its simplest form gives the value of the objective function, the levels of the decision variables, and the amount of slack or surplus in the constraints.

Shadow prices on the constraints

- Solving a linear program usually provides more information about an optimal solution than merely the values of the decision variables.
- Associated with an optimal solution are shadow prices for the constraints (also referred to as dual variables, marginal values, or π values).
- The shadow price on a particular constraint represents *the change in the value of the objective function per unit increase in the righthand-side value of that constraint*.

Shadow prices on the constraints

For example, suppose that the constraint on resource 1 was increased from 60 to 61. What is the change in the value of the objective function from such an increase?

$$\begin{aligned} 6x_1 + \ \ 5x_2 &= \ \ 61, \\ 10x_1 + 20x_2 &= 150, \end{aligned} \quad \rightarrow \quad x_1 = 6\tfrac{5}{7} \text{ and } x_2 = 4\tfrac{1}{7},$$

$$\rightarrow z = 500x_1 + 450x_2 = 500(6\tfrac{5}{7}) + 450(4\tfrac{1}{7}) = 5{,}221\tfrac{3}{7}.$$

$$\rightarrow 5221\tfrac{3}{7} - 5142\tfrac{6}{7} = 78\tfrac{4}{7}.$$

Note on terminology: reduced costs

Shadow prices associated with the non-negativity constraints ($x_i \geq 0$) often are called the reduced costs and usually are reported separately from the shadow prices on the other constraints.

However, they have the identical interpretation.

Objective and Righthand-Side Ranges

- The data for a linear program may not be known with certainty or may be subject to change.
- When solving linear programs, then, it is natural to ask about the sensitivity of the optimal solution to variations in the data.
- For example, over what range can a particular objective-function coefficient vary without changing the optimal solution?

Changes in the Coefficients of the Objective Function

- We will consider first the question of making one-at-a-time changes in the coefficients of the objective function.
- Consider an LP with 2 decision variables. Suppose we consider the contribution per unit of activity 1, and determine the range for that coefficient such that the optimal solution remains unchanged.

Recall:

$$\text{Maximize } z = 500x_1 + 450x_2,$$

subject to:

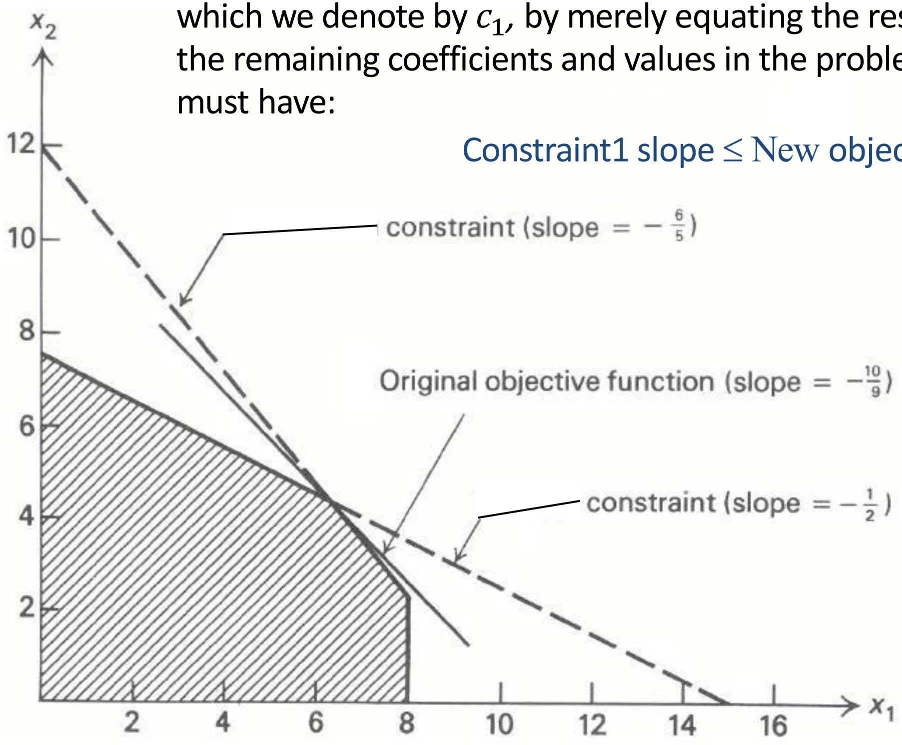
$$\begin{aligned} 6x_1 + 5x_2 &\leq 60 \\ 10x_1 + 20x_2 &\leq 150 \\ x_1 &\leq 8 \end{aligned}$$

$$x_1 \geq 0, \quad x_2 \geq 0.$$

Changes in the Coefficients of the Objective Function

We can determine the range on the coefficient of contribution from activity 1, which we denote by c_1, by merely equating the respective slopes. Assuming the remaining coefficients and values in the problem remain unchanged, we must have:

Constraint1 slope ≤ New objective slope ≤ Constraint2 slope



Changes in the Righthand-Side Values of the Constraints

- Throughout our discussion of shadow prices, we assumed that the constraints defining the optimal solution did not change when the values of their righthand sides were varied.
- Over what range can a particular righthand-side value change without changing the shadow prices associated with that constraint?
- How much can resource 1 capacity be increased and still give us an increase of 78 4/7 per unit of increase?

Changes in the Righthand-Side Values of the Constraints

Maximize $z = 500x_1 + 450x_2,$

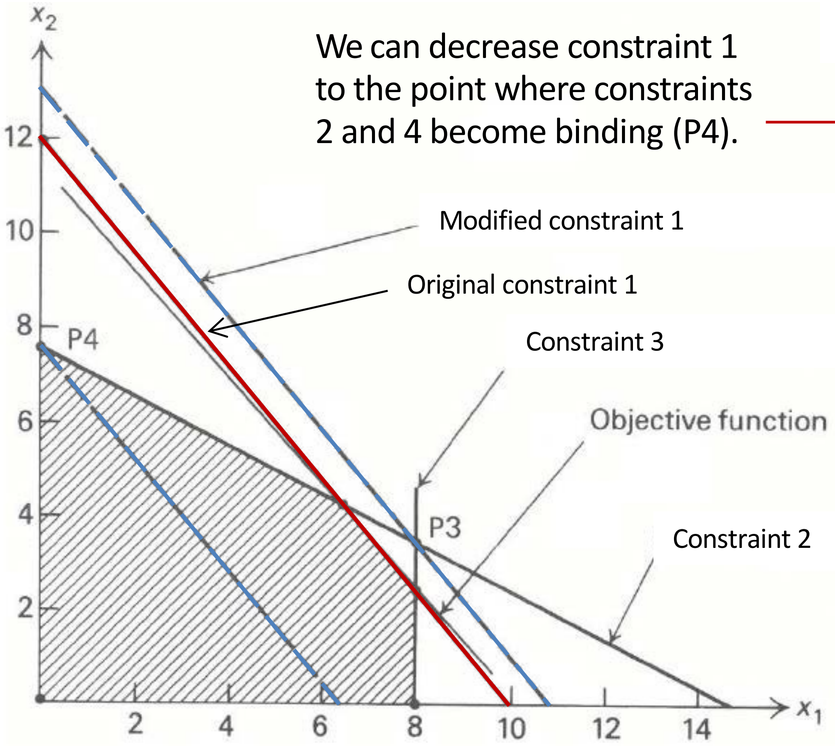
subject to:

$$\begin{aligned} 6x_1 + 5x_2 &\le 60 \\ 10x_1 + 20x_2 &\le 150 \\ x_1 &\le 8 \\ x_1 \ge 0, \quad x_2 &\ge 0. \end{aligned}$$

We can decrease constraint 1 to the point where constraints 2 and 4 become binding (P4).

$$37\tfrac{1}{2} \le b_1 \le 65\tfrac{1}{2},$$

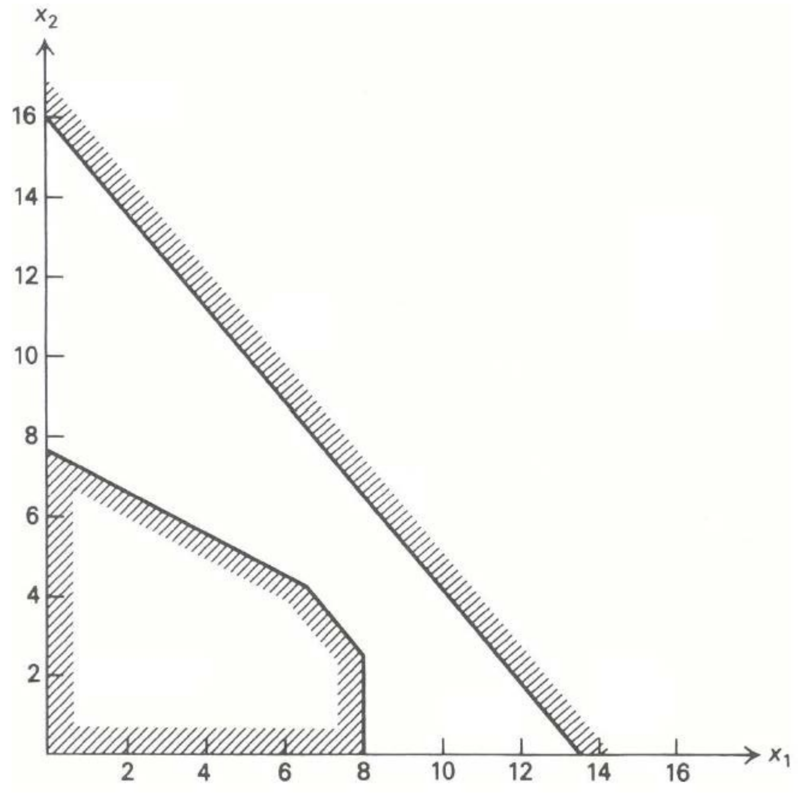
We cannot increase constraint 1 beyond the point where constraints 2 & 3 become binding (P3).



Feasibility

- Until now we have described a number of the properties of an optimal solution to a linear program, assuming 1) that there was such a solution and 2) that we were able to find it.
- It could happen that a linear program has no feasible solution.
- An infeasible linear program might result from a poorly formulated problem, or from a situation where requirements exceed the capacity of the existing available resources.

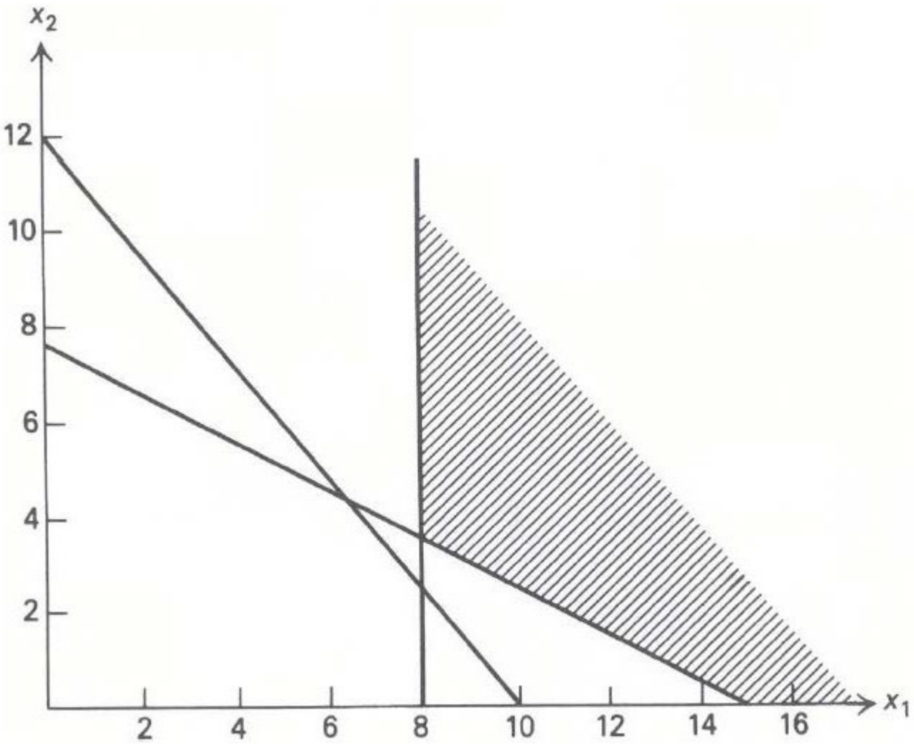
Example: An unfeasible group of constraints



Feasibility

- Until now we have described a number of the properties of an optimal solution to a linear program, assuming 1) that there was such a solution and 2) that we were able to find it.
- It could happen that a linear program has no feasible solution.
- An infeasible linear program might result from a poorly formulated problem, or from a situation where requirements exceed the capacity of the existing available resources.

Example: An unbounded solution



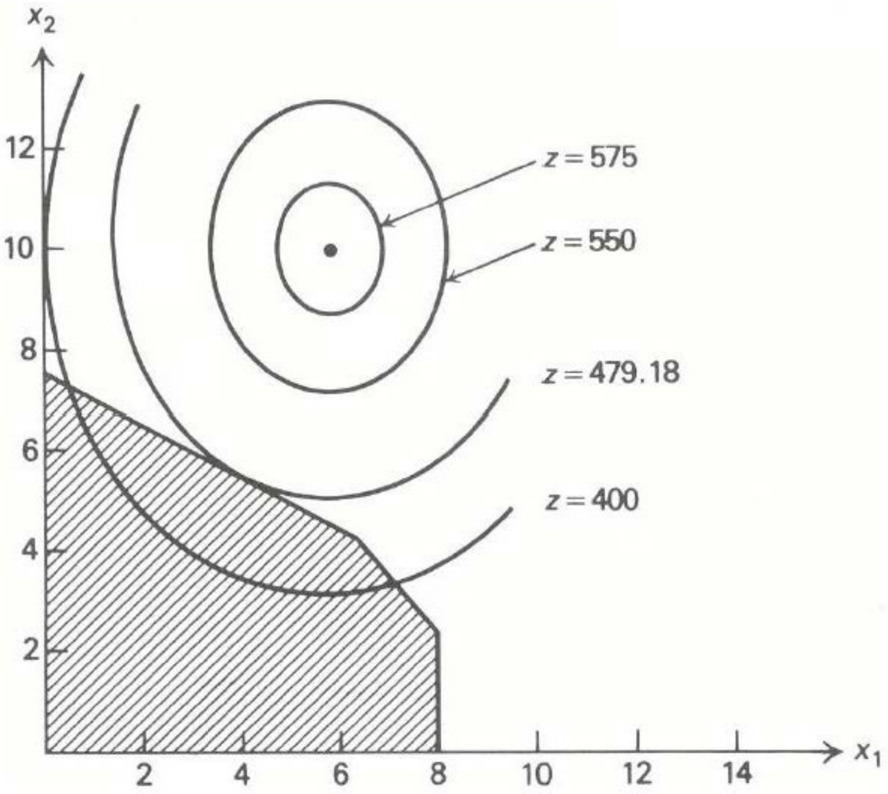
Non-linear programming

$$\text{Maximize } z = 60x_1 - 5x_1^2 + 80x_2 - 4x_2^2,$$

subject to:

$$\begin{aligned} 6x_1 + 5x_2 &\leq 60, \\ 10x_1 + 20x_2 &\leq 150, \\ x_1 &\leq 8, \\ x_1 \geq 0, \quad x_2 &\geq 0. \end{aligned}$$

Non-linear programming



References

1. AMP - Chapter 1 Mathematical Programming: an overview
   Companion slides of Applied Mathematical Programming by Bradley, Hax, and Magnanti (Addison-Wesley, 1977) prepared by José Fernando Oliveira Maria Antónia Carravilla
2. Lecture slides from Prof Carolina Osorio (MIT 1.041)
3. Introduction to Linear Optimization (Bertsimas and Tsitsiklis) available in library
4. Formulating an MP: An Overview (Nathaniel Grier) available through Canvas
5. Introduction to Operations Research (Hillier and Lierberman)