Spring 2024

Simplex method

Solving linear programs

Cathy Wu

1.041/1.200 Transportation: Foundations and Methods

Readings

- Bradley, Stephen P., Arnoldo C. Hax, and Thomas L. Magnanti. **Applied mathematical programming**. Addison-Wesley (1977). Chapter 2 Solving Linear Programs [URL]

Outline

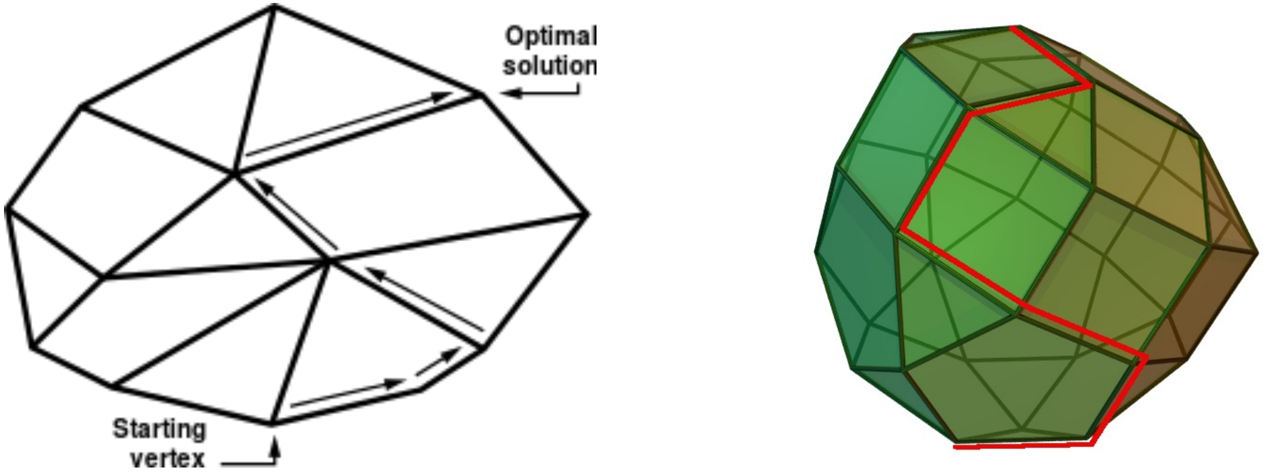
1. Simplex method
2. Reduction to canonical form
3. Trailer production problem

Outline

1. **Simplex method**
    a. Canonical form
    b. Basic and non-basic variables
    c. Pivoting
2. Reduction to canonical form
3. Trailer production problem

The simplex method

- A systematic procedure for solving linear programs
- Proceeds by moving from one feasible solution to another, at each step improving the value of the objective function.
- Terminates after a finite number of such transitions.



A generalization of geometric solutions (L20) to high dimensions.

Images: Wikipedia – Simplex Algorithm

The simplex method

- Two important characteristics of the simplex method:
  - The method is robust.
    - It solves any linear program;
    - It detects redundant constraints in the problem formulation;
    - It identifies instances when the objective value is unbounded over the feasible region; and
    - It solves problems with one or more optimal solutions.
    - The method is also self-initiating.
      - It uses itself either to generate an appropriate feasible solution, as required, to start the method, or to show that the problem has no feasible solution.
  - The simplex method provides much more than just optimal solutions.
    - Recall L20: It indicates how the optimal solution varies as a function of the problem data (cost coefficients, constraint coefficients, and righthand-side data).
    - Information intimately related to a linear program called the "dual" to the given problem: the simplex method automatically solves this dual problem along with the given problem.

The canonical form

Maximize $z = 0x_1 + 0x_2 - 3x_3 - x_4 + 20,$

Objective function 1

subject to:

$$\begin{array}{lll} x_1 \qquad\qquad - 3x_3 + 3x_4 = 6 & & (1) \\ \qquad\quad x_2 - 8x_3 + 4x_4 = 4 & & (2) \\ x_j \geq 0 \qquad (j = 1, 2, 3, 4) & & \end{array}$$

Any linear programming problem
can be transformed so
that it is in canonical form!

1. All decision variables are constrained to be nonnegative.
2. All constraints, except for the nonnegativity of decision variables, are stated as equalities.
3. The righthand-side coefficients are all nonnegative.
4. **One decision variable is isolated in each constraint with a +1 coefficient** (x_1 in constraint (1) and x_2 in constraint (2)). The variable isolated in a given constraint does not appear in any other constraint and appears with a zero coefficient in the objective function.

$\Big\}$ Recall: Standard form

Discussion

Maximize $z = 0x_1 + 0x_2 - 3x_3 - x_4 + 20$,
subject to:

$$\begin{aligned} x_1 \qquad\quad - 3x_3 + 3x_4 &= 6 \\ x_2 - 8x_3 + 4x_4 &= 4 \\ x_j \geq 0 \qquad (j = 1, 2, 3, 4)& \end{aligned}$$

- Given any values for x_3 and x_4, the values of x_1 and x_2 are determined uniquely by the equalities.
  - In fact, setting $x_3 = x_4 = 0$ immediately gives a feasible solution with $x_1 = 6$ and $x_2 = 4$.
  - Solutions such as these will play a central role in the simplex method and are referred to as basic feasible solutions.
- In general, given a canonical form for any linear program, a basic feasible solution is given by setting the variable isolated in constraint j, called the jth basic-variable, equal to the righthand side of the jth constraint and by setting the remaining variables, called nonbasic, all to zero.
- Collectively the basic variables are termed a basis.

Maximize $z = 0x_1 + 0x_2 - 3x_3 - x_4 + 20$,
subject to:

$$\begin{aligned} x_1 \qquad\quad - 3x_3 + 3x_4 &= 6 \\ x_2 - 8x_3 + 4x_4 &= 4 \\ x_j \geq 0 \qquad (j = 1, 2, 3, 4)& \end{aligned}$$

Discussion

- In the example above, the basic feasible solution $x_1 = 6, x_2 = 4, x_3 = 0, x_4 = 0$ is optimal.
  - For any other feasible solution, x_3 and x_4 must remain nonnegative.
  - Since their coefficients in the objective function are negative, if either x_3 or x_4 is positive, z will be less than 20.
  - Thus, the maximum value for z is obtained when $x_3 = x_4 = 0$.

Optimality Criterion

Maximize $z = 0x_1 + 0x_2 - 3x_3 - x_4 + 20$,
subject to:

$$\begin{aligned} x_1 \qquad\quad - 3x_3 + 3x_4 &= 6 \\ x_2 - 8x_3 + 4x_4 &= 4 \\ x_j \geq 0 \qquad (j = 1, 2, 3, 4) \end{aligned}$$

- Suppose that, in a maximization problem, every nonbasic variable has a nonpositive coefficient in the objective function of a canonical form.
- Then the basic feasible solution given by the canonical form maximizes the objective function over the feasible region.

Unbounded Objective Value

Maximize $z = 0x_1 + 0x_2 + 3x_3 - x_4 + 20,$

subject to:

Objective function 2

$$x_1 \qquad - 3x_3 + 3x_4 = 6 \qquad (1)$$

$$x_2 - 8x_3 + 4x_4 = 4 \qquad (2)$$

$$x_j \geq 0 \qquad (j = 1, 2, 3, 4)$$

- Since x_3 now has a positive coefficient in the objective function, it appears promising to increase the value of x_3 as much as possible.
- Let us maintain $x_4 = 0$, increase x_3 to a value t to be determined, and update x_1 and x_2 to preserve feasibility.

Discussion

Maximize $z = 0x_1 + 0x_2 + 3x_3 - x_4 + 20$,
subject to:

$$\begin{array}{rl} x_1 \qquad\quad - 3x_3 + 3x_4 &= 6 \\ x_2 - 8x_3 + 4x_4 &= 4 \\ x_j \geq 0 \qquad (j = 1, 2, 3, 4) & \end{array}$$

$$\begin{array}{r} x_1 \qquad\quad - 3x_3 + 3x_4 = 6 \\ x_2 - 8x_3 + 4x_4 = 4 \\ z = 0x_1 + 0x_2 + 3x_3 - x_4 + 20 \end{array} \longrightarrow \begin{array}{l} x_1 = 6 + 3t \\ x_2 = 4 + 8t \\ z = 20 + 3t \end{array}$$

- No matter how large t becomes, x_1 and x_2 remain nonnegative. In fact, as t approaches $+\infty$, z approaches $+\infty$.
- In this case, the objective function is unbounded over the feasible region.

Unboundedness Criterion

Maximize $z = 0x_1 + 0x_2 + 3x_3 - x_4 + 20$,
subject to:

$$\begin{aligned} x_1 \qquad\qquad - 3x_3 + 3x_4 &= 6 \\ x_2 - 8x_3 + 4x_4 &= 4 \\ x_j \geq 0 \qquad (j = 1, 2, 3, 4)& \end{aligned}$$

- Suppose that, in a maximization problem, some nonbasic variable has a positive coefficient in the objective function of a canonical form.
- If that variable has negative or zero coefficients in all constraints, then the objective function is unbounded from above over the feasible region.

Improving a Nonoptimal Solution

Maximize $z = 0x_1 + 0x_2 - 3x_3 + x_4 + 20$,
subject to:

$$\begin{aligned} x_1 \qquad\qquad - 3x_3 + 3x_4 &= 6 \\ x_2 - 8x_3 + 4x_4 &= 4 \\ x_j \geq 0 \qquad (j = 1,2,3,4) & \end{aligned}$$

Objective function 3

- As x_4 increases, z increases.
- Maintaining $x_3 = 0$, let us increase x_4 to a value t, and update x_1 and x_2 to preserve feasibility.

Discussion

Maximize $z = 0x_1 + 0x_2 - 3x_3 + x_4 + 20$, subject to:

$$\begin{aligned} x_1 \qquad\quad - 3x_3 + 3x_4 &= 6 \\ x_2 - 8x_3 + 4x_4 &= 4 \\ x_j \geq 0 \qquad (j = 1, 2, 3, 4)& \end{aligned}$$

$$\begin{aligned} x_1 \qquad\quad - 3x_3 + 3x_4 &= 6 \\ x_2 - 8x_3 + 4x_4 &= 4 \\ z = 0x_1 + 0x_2 - 3x_3 + x_4 + 20& \end{aligned} \quad \longrightarrow \quad \begin{aligned} x_1 &= 6 - 3t \\ x_2 &= 4 - 4t \\ z &= 20 + t \end{aligned}$$

- If x_1 and x_2 are to remain nonnegative, we require:

$$6 - 3t \geq 0, \quad \text{that is, } t \leq \frac{6}{3} = 2$$

  and

$$4 - 4t \geq 0, \quad \text{that is, } t \leq \frac{4}{4} = 1$$

- Therefore, the largest value for t that maintains a feasible solution is $t = 1$.
- When $t = 1$, the new solution becomes $x_1 = 3, x_2 = 0, x_3 = 0, x_4 = 1$, which has an associated value of $z = 21$ in the objective function.

Discussion

Maximize $z = 0x_1 + 0x_2 - 3x_3 + x_4 + 20,$
subject to:

$$\begin{aligned} x_1 \qquad\qquad - 3x_3 + 3x_4 &= 6 \\ x_2 - 8x_3 + 4x_4 &= 4 \\ x_j \geq 0 \qquad (j = 1, 2, 3, 4)& \end{aligned}$$

- Note that, in the new solution, x_4 has a positive value and x_2 has become zero.
- This is a fundamental part of "pivoting" from corner of the solution polyhedron to another.
- Since nonbasic variables were previously given zero values before, it appears that x_4 has replaced x_2 as a basic variable.
- In fact, it is fairly simple to manipulate Eqs. (1) and (2) algebraically to produce a new canonical form, where x_1 and x_4 become the basic variables.

Discussion

- If x_4 is to become a basic variable, it should appear with coefficient +1 in Eq. (2), and with zero coefficients in Eq. (1) and in the objective function.
- To obtain a +1 coefficient in Eq. (2), we divide that equation by 4.

$$\begin{array}{ll} (1) & x_1 - 3x_3 + 3x_4 = 6 \\ (2) & x_2 - 8x_3 + 4x_4 = 4 \end{array} \longrightarrow \begin{array}{l} x_1 - 3x_3 + 3x_4 = 6 \\ \frac{1}{4}x_2 - 2x_3 + x_4 = 1 \end{array}$$

Discussion

- To eliminate x_4 from the first constraint, we may multiply Eq. (2’) by 3 and subtract it from constraint (1).

$$\begin{array}{ll} (1) & x_1 \qquad\qquad - 3x_3 + 3x_4 = 6 \\ (2') & \qquad \frac{1}{4}x_2 - 2x_3 + \ x_4 = 1 \end{array} \quad \rightarrow \quad x_1 - \frac{3}{4}x_2 + 3x_3 = 3$$

- We may rearrange the objective function and write it as:

$$(-z) - 3x_3 + x_4 = -20$$

and use the same technique to eliminate x_4; that is, multiply (2’) by 1 and subtract it from the above:

$$(-z) - \frac{1}{4}x_2 - x_3 = -21$$

The new global system becomes

Maximize $z = 0x_1 - \frac{1}{4}x_2 - x_3 + 0x_4 + 21$,

subject to:

$$
\begin{aligned}
x_1 - \frac{3}{4}x_2 - 3x_3 \qquad &= 3 \\
\frac{1}{4}x_2 - 2x_3 + x_4 &= 1 \\
x_j \geq 0 \qquad (j = 1, 2, 3, 4)
\end{aligned}
$$

This procedure for generating a new basic variable is called *pivoting*

- Now the problem is in canonical form with x_1 and x_4 as basic variables, and z has increased from 20 to 21.
- Consequently, we are in a position to reapply the arguments of this section, beginning with this improved solution.
- However, in this case, the new canonical form satisfies the optimality criterion since all nonbasic variables have nonpositive coefficients in the objective function, and thus the basic feasible solution $x_1 = 3, x_2 = 0, x_3 = 0, x_4 = 1$, is optimal.

Improvement Criterion

- Suppose that, in a maximization problem, some nonbasic variable has a positive coefficient in the objective function of a canonical form.
- If that variable has a positive coefficient in some constraint, then a new basic feasible solution may be obtained by pivoting.

Discussion

$$\begin{aligned} x_1 \qquad\qquad - 3x_3 + 3x_4 &= 6 \\ x_2 - 8x_3 + 4x_4 &= 4 \\ z = 0x_1 + 0x_2 - 3x_3 + x_4 &+ 20 \end{aligned}$$

- Recall that we chose the constraint to pivot in (and consequently the variable to drop from the basis) by determining which basic variable first goes to zero as we increase the nonbasic variable x_4.
- The constraint is selected by taking the ratio of the righthand-side coefficients to the coefficients of x_4 in the constraints, i.e., by performing the ratio test:

$$\min\left\{\frac{6}{3}, \frac{4}{4}\right\}$$

Discussion

- Note, however, that if the coefficient of x_4 in the second constraint were −4 instead of +4, the values for x_1 and x_2 would be given by:

$$\begin{aligned} x_1 \qquad\quad - 3x_3 + 3x_4 &= 6 \\ x_2 - 8x_3 - 4x_4 &= 4 \end{aligned} \quad \rightarrow \quad \begin{aligned} x_1 &= 6 - 3t \\ x_2 &= 4 + 4t \end{aligned}$$

  so that as $x_4 = t$ increases from 0, x_2 never becomes zero. In this case, we would increase x_4 to $t = \frac{6}{3} = 2$.

- This observation applies in general for any number of constraints, so that we need never compute ratios for nonpositive coefficients of the variable that is coming into the basis.

Ratio and Pivoting Criterion

- When improving a given canonical form by introducing variable x_s into the basis, pivot in a constraint that gives the minimum ratio of righthand-side coefficient to corresponding x_s coefficient.
- Compute these ratios only for constraints that have a positive coefficient for x_s.

Outline

1. Simplex method
2. **Reduction to canonical form**
3. Trailer production problem

Reduction to Canonical Form

- To this point we have been solving linear programs posed in canonical form with
  1. nonnegative variables,
  2. equality constraints,
  3. nonnegative righthand-side coefficients, and
  4. one basic variable isolated in each constraint.
- We will now recall how to transform any linear program to this canonical form.

Inequality constraints

$$40x_1 + 10x_2 + 6x_3 \leq 55.0,$$
$$40x_1 + 10x_2 + 6x_3 \geq 32.5$$

- Introduce two new nonnegative variables:
- x_5 measures the amount that the consumption of resource falls short of the maximum available, and is called a slack variable;
- x_6 is the amount of product in excess of the minimum requirement and is called a surplus variable.

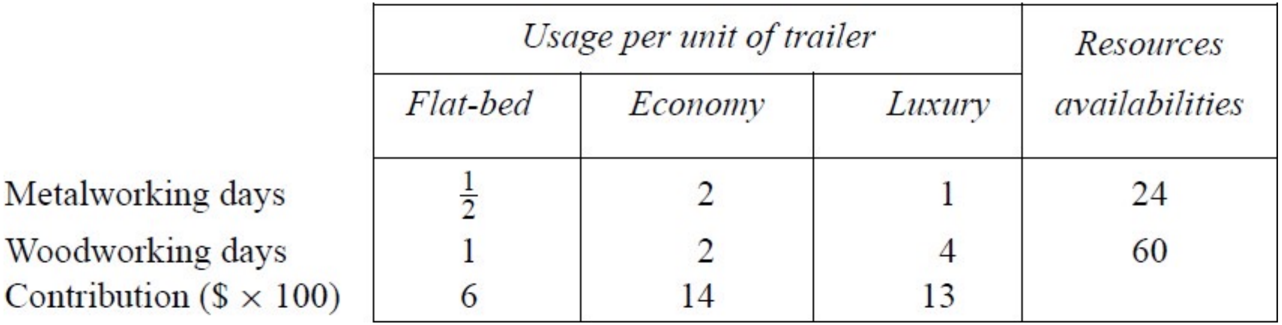
$$40x_1 + 10x_2 + 6x_3 + x_5 = 55.0,$$
$$40x_1 + 10x_2 + 6x_3 - x_6 = 32.5$$

Outline

1. Simplex method
2. Reduction to canonical form
3. **Trailer production problem**

Trailer production problem

- The owner of a shop producing automobile trailers wishes to determine the best mix for his three products:
  - flat-bed trailers
  - economy trailers
  - luxury trailers.
- His shop is limited to working 24 days/month on metalworking and 60 days/month on woodworking for these products. The following table indicates production data for the trailers.



	Usage per unit of trailer			*Resources availabilities*
	Flat-bed	*Economy*	*Luxury*	
Metalworking days	$\frac{1}{2}$	2	1	24
Woodworking days	1	2	4	60
Contribution ($ × 100)	6	14	13	

LP Model

- Let the decision variables of the problem be:
  - x_1 = Number of flat-bed trailers produced per month
  - x_2 = Number of economy trailers produced per month
  - x_3 = Number of luxury trailers produced per month

Maximize $z = 6x_1 + 14x_2 + 13x_3$,

subject to:

$$\frac{1}{2}x_1 + 2x_2 + x_3 \leq 24$$

$$x_1 + 2x_2 + 4x_3 \leq 60$$

$$x_1 \geq 0, \qquad x_2 \geq 0, \qquad x_3 \geq 0$$

Canonical form

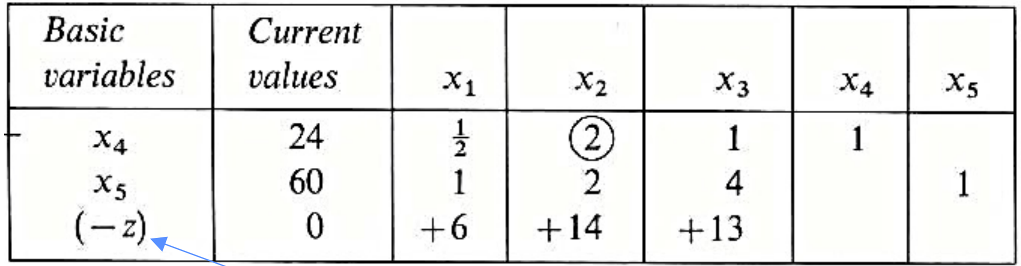
Maximize $z = 6x_1 + 14x_2 + 13x_3,$

subject to:

$$\frac{1}{2}x_1 + 2x_2 + x_3 + x_4 = 24$$

$$x_1 + 2x_2 + 4x_3 + x_5 = 60$$

$$x_j \geq 0 \quad (j = 1, 2, 3, 4, 5)$$



Basic variables	*Current values*	x_1	x_2	x_3	x_4	x_5
x_4	24	$\frac{1}{2}$	②	1	1	
x_5	60	1	2	4		1
$(-z)$	0	+6	+14	+13		

$$(-z) + 6x_1 + 14x_2 + 13x_3 = 0.$$

Iterations

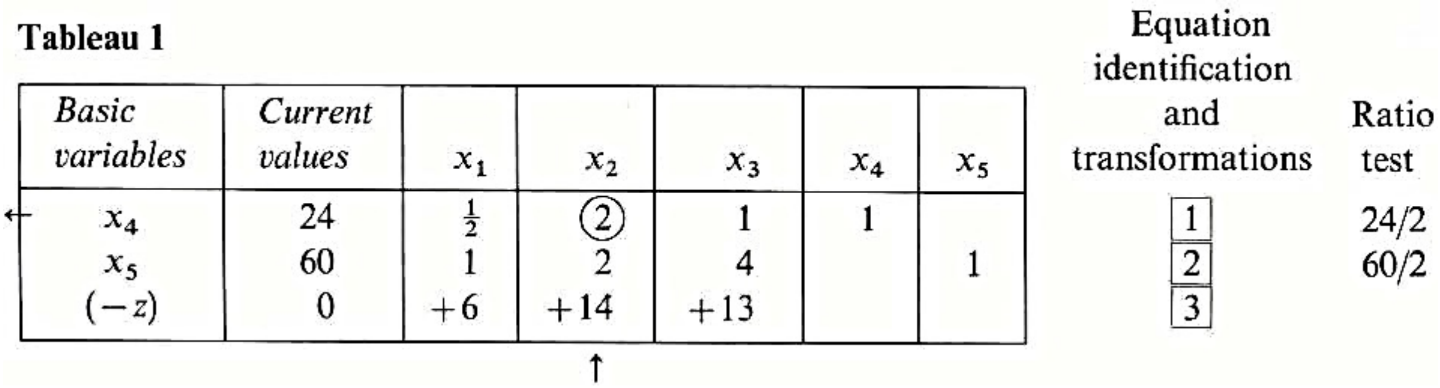


Tableau 1

	Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	Equation identification and transformations	Ratio test
←	x_4	24	$\frac{1}{2}$	②	1	1		$\boxed{1}$	24/2
	x_5	60	1	2	4		1	$\boxed{2}$	60/2
	$(-z)$	0	+6	+14	+13			$\boxed{3}$	
				↑					

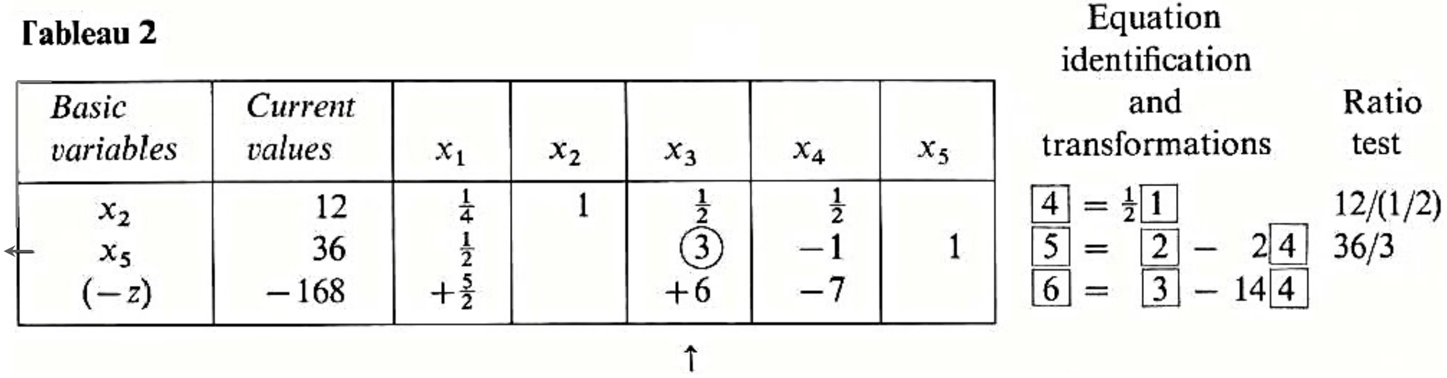


Tableau 2

	Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	Equation identification and transformations	Ratio test
	x_2	12	$\frac{1}{4}$	1	$\frac{1}{2}$	$\frac{1}{2}$		$\boxed{4} = \frac{1}{2}\boxed{1}$	12/(1/2)
←	x_5	36	$\frac{1}{2}$		③	-1	1	$\boxed{5} = \boxed{2} - 2\boxed{4}$	36/3
	$(-z)$	-168	$+\frac{5}{2}$		+6	-7		$\boxed{6} = \boxed{3} - 14\boxed{4}$	
					↑				

Iterations

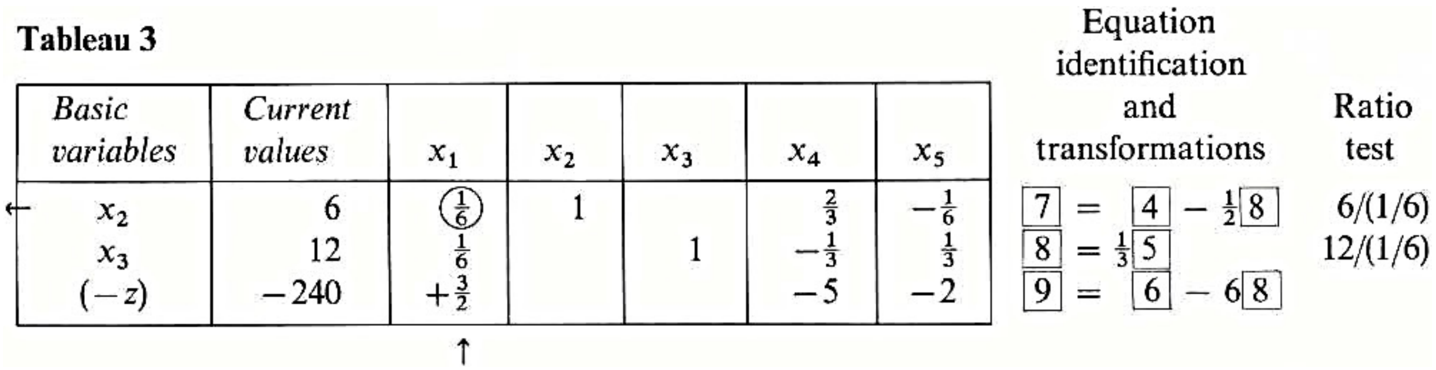


Tableau 3

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	Equation identification and transformations	Ratio test
← x_2	6	$\textcircled{\frac{1}{6}}$	1		$\frac{2}{3}$	$-\frac{1}{6}$	$\boxed{7} = \boxed{4} - \frac{1}{2}\boxed{8}$	6/(1/6)
x_3	12	$\frac{1}{6}$		1	$-\frac{1}{3}$	$\frac{1}{3}$	$\boxed{8} = \frac{1}{3}\boxed{5}$	12/(1/6)
$(-z)$	-240	$+\frac{3}{2}$			-5	-2	$\boxed{9} = \boxed{6} - 6\boxed{8}$	
		↑						

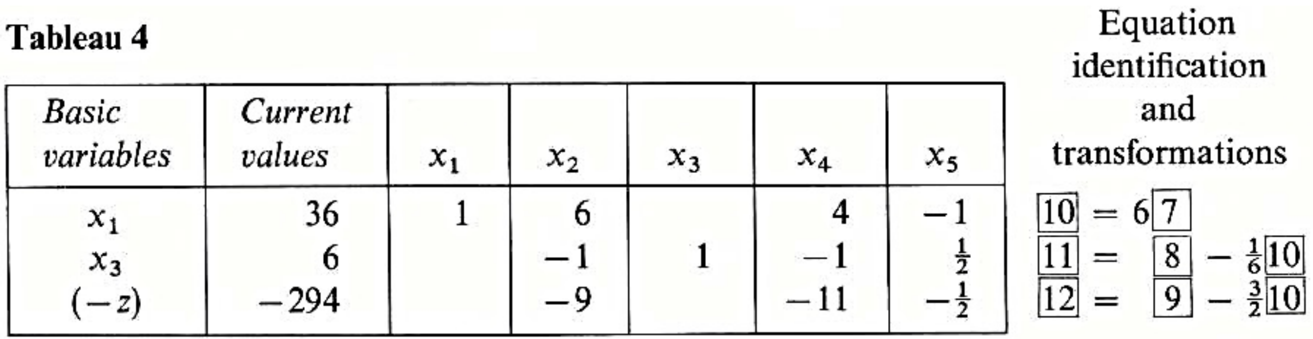


Tableau 4

Basic variables	Current values	x_1	x_2	x_3	x_4	x_5	Equation identification and transformations
x_1	36	1	6		4	-1	$\boxed{10} = 6\boxed{7}$
x_3	6		-1	1	-1	$\frac{1}{2}$	$\boxed{11} = \boxed{8} - \frac{1}{6}\boxed{10}$
$(-z)$	-294		-9		-11	$-\frac{1}{2}$	$\boxed{12} = \boxed{9} - \frac{3}{2}\boxed{10}$

Minimization problems

- Enters the basis the nonbasic variable that has a negative coefficient in the objective function of a canonical form.
- The solution is optimal when every nonbasic variable has a nonnegative coefficient in the objective function of a canonical form.

Formal Procedure

Simplex Algorithm (Maximization Form)

0. The problem is initially in canonical form and all $\bar{b}_i \geq 0$.
1. If $\bar{c}_j \leq 0$ for $j = 1, 2, \ldots, n$, then *stop*; we are optimal. If we continue then there exists some $\bar{c}_j > 0$.
2. Choose the column to pivot in (i.e. the variable to introduce into the basis) by:

$$\bar{c}_s = \max_j \left\{ \bar{c}_j \middle| \bar{c}_j > 0 \right\}$$

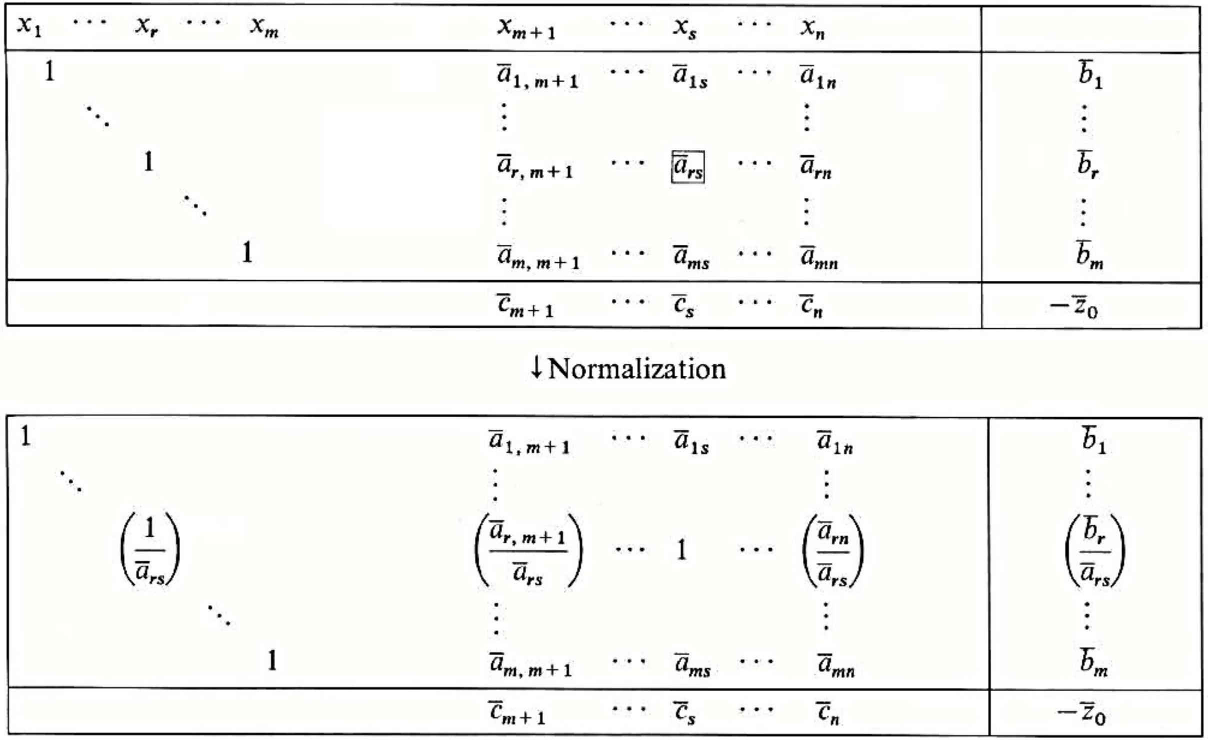
   If $\bar{a}_{is} \leq 0$ for $i = 1, 2, \ldots, m$, then *stop*; the primal problem is unbounded. If we continue, then $\bar{a}_{is} > 0$ for some $i = 1, 2, \ldots, m$.

3. Choose row r to pivot in (i.e. the variable to drop from the basis) by the ratio test:

$$\frac{\bar{b}_r}{\bar{a}_{rs}} \min_i \left\{ \frac{\bar{b}_i}{\bar{a}_{is}} \middle| \bar{a}_{is} > 0 \right\}$$

4. Replace the basic variable in row r with variable s and reestablish the canonical form (i.e. pivot on the coefficient $\bar{a}_{rs}$).
5. Go to step (1).

STEP (4) Pivoting

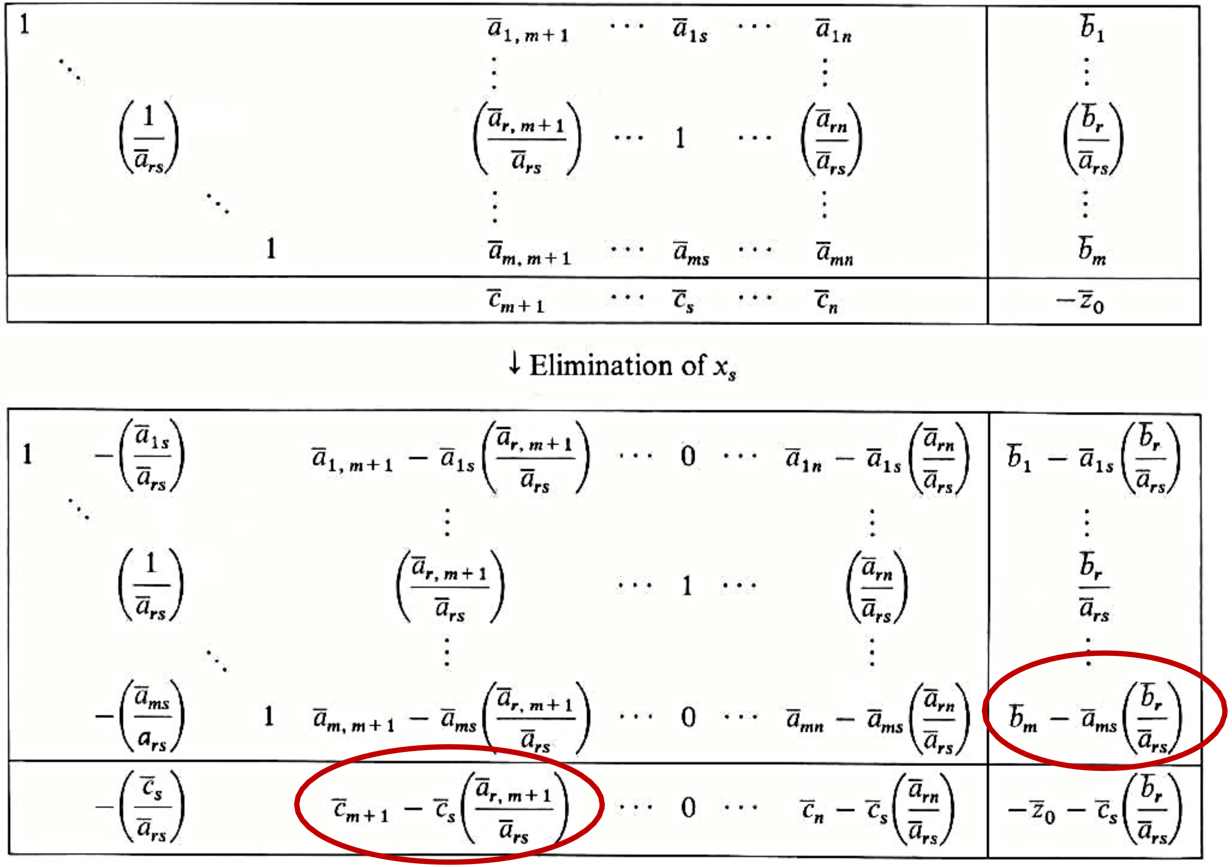


$$\begin{array}{cccccccccc|c}
x_1 & \cdots & x_r & \cdots & x_m & x_{m+1} & \cdots & x_s & \cdots & x_n & \\ \hline
1 & & & & & \bar{a}_{1,m+1} & \cdots & \bar{a}_{1s} & \cdots & \bar{a}_{1n} & \bar{b}_1 \\
 & \ddots & & & & \vdots & & & & \vdots & \vdots \\
 & & 1 & & & \bar{a}_{r,m+1} & \cdots & \boxed{\bar{a}_{rs}} & \cdots & \bar{a}_{rn} & \bar{b}_r \\
 & & & \ddots & & \vdots & & & & \vdots & \vdots \\
 & & & & 1 & \bar{a}_{m,m+1} & \cdots & \bar{a}_{ms} & \cdots & \bar{a}_{mn} & \bar{b}_m \\ \hline
 & & & & & \bar{c}_{m+1} & \cdots & \bar{c}_s & \cdots & \bar{c}_n & -\bar{z}_0
\end{array}$$

↓Normalization

$$\begin{array}{cccccccccc|c}
1 & & & & & \bar{a}_{1,m+1} & \cdots & \bar{a}_{1s} & \cdots & \bar{a}_{1n} & \bar{b}_1 \\
 & \ddots & & & & \vdots & & & & \vdots & \vdots \\
 & & \left(\frac{1}{\bar{a}_{rs}}\right) & & & \left(\frac{\bar{a}_{r,m+1}}{\bar{a}_{rs}}\right) & \cdots & 1 & \cdots & \left(\frac{\bar{a}_{rn}}{\bar{a}_{rs}}\right) & \left(\frac{\bar{b}_r}{\bar{a}_{rs}}\right) \\
 & & & \ddots & & \vdots & & & & \vdots & \vdots \\
 & & & & 1 & \bar{a}_{m,m+1} & \cdots & \bar{a}_{ms} & \cdots & \bar{a}_{mn} & \bar{b}_m \\ \hline
 & & & & & \bar{c}_{m+1} & \cdots & \bar{c}_s & \cdots & \bar{c}_n & -\bar{z}_0
\end{array}$$

STEP (4) Pivoting



$$
\begin{array}{cccccccccc|c}
1 & & & & & \bar{a}_{1,m+1} & \cdots & \bar{a}_{1s} & \cdots & \bar{a}_{1n} & \bar{b}_1 \\
& \ddots & & & & \vdots & & & & \vdots & \vdots \\
& & \left(\dfrac{1}{\bar{a}_{rs}}\right) & & & \left(\dfrac{\bar{a}_{r,m+1}}{\bar{a}_{rs}}\right) & \cdots & 1 & \cdots & \left(\dfrac{\bar{a}_{rn}}{\bar{a}_{rs}}\right) & \left(\dfrac{\bar{b}_r}{\bar{a}_{rs}}\right) \\
& & & \ddots & & \vdots & & & & \vdots & \vdots \\
& & & & 1 & \bar{a}_{m,m+1} & \cdots & \bar{a}_{ms} & \cdots & \bar{a}_{mn} & \bar{b}_m \\
\hline
& & & & & \bar{c}_{m+1} & \cdots & \bar{c}_s & \cdots & \bar{c}_n & -\bar{z}_0
\end{array}
$$

↓ Elimination of x_s

$$
\begin{array}{cccccccccc|c}
1 & & -\left(\dfrac{\bar{a}_{1s}}{\bar{a}_{rs}}\right) & & & \bar{a}_{1,m+1} - \bar{a}_{1s}\left(\dfrac{\bar{a}_{r,m+1}}{\bar{a}_{rs}}\right) & \cdots & 0 & \cdots & \bar{a}_{1n} - \bar{a}_{1s}\left(\dfrac{\bar{a}_{rn}}{\bar{a}_{rs}}\right) & \bar{b}_1 - \bar{a}_{1s}\left(\dfrac{\bar{b}_r}{\bar{a}_{rs}}\right) \\
& \ddots & & & & \vdots & & & & \vdots & \vdots \\
& & \left(\dfrac{1}{\bar{a}_{rs}}\right) & & & \left(\dfrac{\bar{a}_{r,m+1}}{\bar{a}_{rs}}\right) & \cdots & 1 & \cdots & \left(\dfrac{\bar{a}_{rn}}{\bar{a}_{rs}}\right) & \dfrac{\bar{b}_r}{\bar{a}_{rs}} \\
& & & \ddots & & \vdots & & & & \vdots & \vdots \\
& & -\left(\dfrac{\bar{a}_{ms}}{\bar{a}_{rs}}\right) & & 1 & \bar{a}_{m,m+1} - \bar{a}_{ms}\left(\dfrac{\bar{a}_{r,m+1}}{\bar{a}_{rs}}\right) & \cdots & 0 & \cdots & \bar{a}_{mn} - \bar{a}_{ms}\left(\dfrac{\bar{a}_{rn}}{\bar{a}_{rs}}\right) & \boxed{\bar{b}_m - \bar{a}_{ms}\left(\dfrac{\bar{b}_r}{\bar{a}_{rs}}\right)} \\
\hline
& & -\left(\dfrac{\bar{c}_s}{\bar{a}_{rs}}\right) & & & \boxed{\bar{c}_{m+1} - \bar{c}_s\left(\dfrac{\bar{a}_{r,m+1}}{\bar{a}_{rs}}\right)} & \cdots & 0 & \cdots & \bar{c}_n - \bar{c}_s\left(\dfrac{\bar{a}_{rn}}{\bar{a}_{rs}}\right) & -\bar{z}_0 - \bar{c}_s\left(\dfrac{\bar{b}_r}{\bar{a}_{rs}}\right)
\end{array}
$$