Uncertainty in transportation

Probabilistic Concepts

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1.041/1.200 Transportation: Foundations and Methods

Readings

1. Larson, Richard C. and Amedeo R. Odoni. **Urban Operations Research**. Prentice-Hall (1981). Chapter 2: Probability. URL.

1.041/1.200 overview

Model transportation systems

Optimize transportation systems

Unit 1

Unit 2

Unit 3

Unit 4

Deterministic

Stochastic

Multi-stage

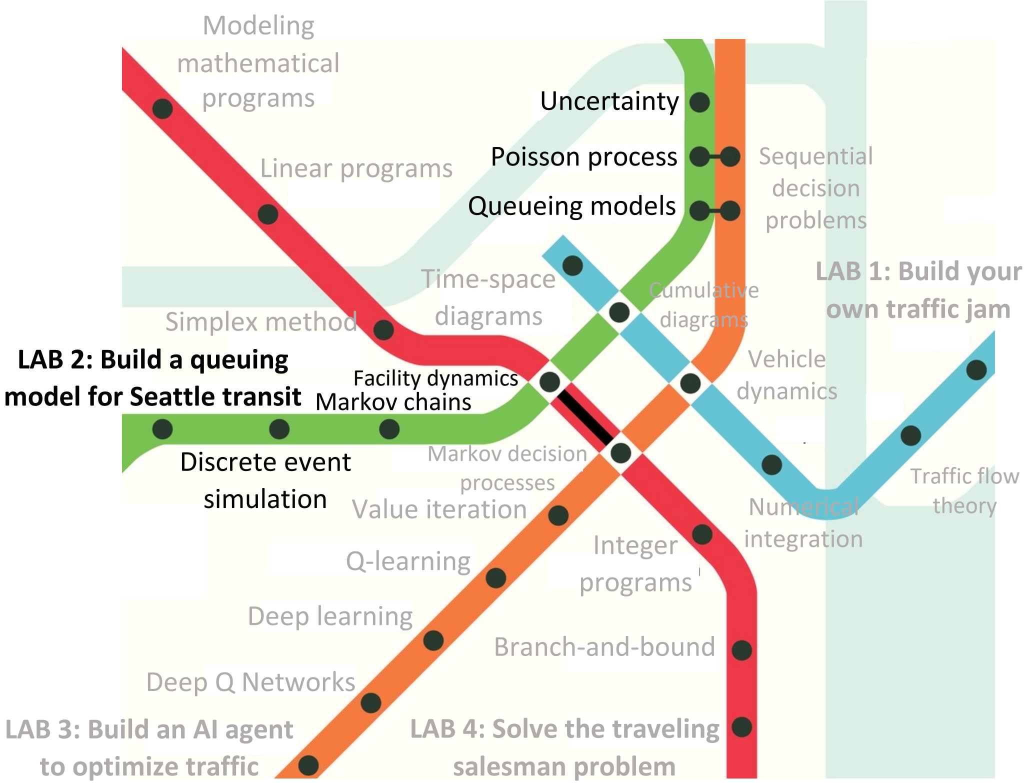
Single-stage

Unit 2: Queuing systems

Unit 2

Modeling

Stochastic



Outline

1. Uncertainty in transportation
2. Expected waiting time of a bus passenger problem

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1. **Uncertainty in transportation**
2. Expected waiting time of a bus passenger problem

Uncertainty

- Decision-making typically takes place under uncertainty

- Today: deterministic → probabilistic concepts.
- What is the role of probability for modeling in transportation?

Examples: Sources of uncertainty in transportation

- Road traffic
  - Weather, road condition
  - Local events
  - Time of day (demands)
  - Day of week (demands)
  - Types of drivers on the road
  - Exact number of vehicles, driving behavior
- Bus stop
  - Exact arrivals of passengers
  - Road traffic
- Uber/Lyft ride
  - Availability of drivers
  - Other users making requests
  - Road traffic
  - Exact traffic signal timing
- Airline departure
  - Engine malfunction, safety checks
  - Boarding time
  - Run-way availability

Mahmassani, H. S. (1984). Uncertainty in transportation systems evaluation: issues and approaches. *Transportation planning and technology*, 9(1):1–12.

Empirical (cumulative) distribution function

- From relative frequencies:
  - Example: k = time headway (minutes) of bus arrivals

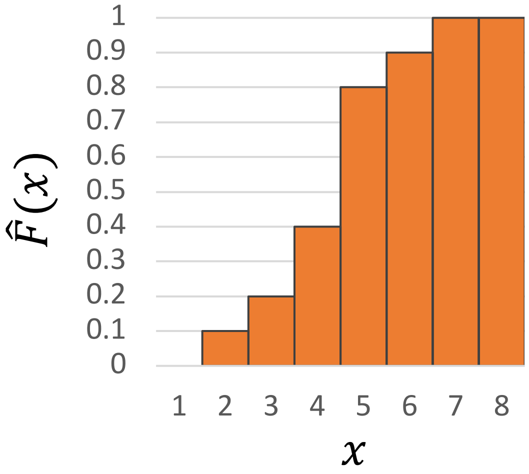
k	2	3	4	5	6	7
$v_k = \frac{n_k}{n}$	.1	.1	.2	.4	.1	.1

- Compute cumulative frequencies:

k	2	3	4	5	6	7
c_k	.1	.2	.4	.8	.9	1

- Yielding empirical cumulative distribution function (cdf):

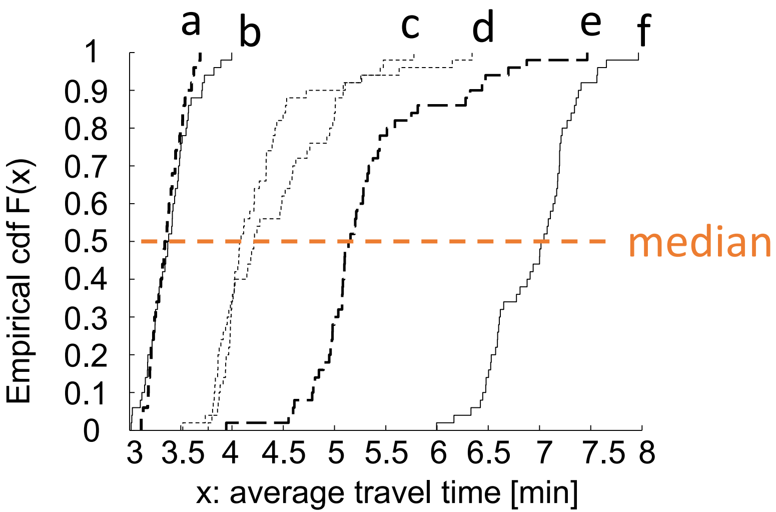
$$\hat{F}(x)\begin{cases} 0 & : \quad -\infty < x < 2 \\ c_k & : \quad k \le x < k+1, k = 2, \dots 6 \\ 1 & : \quad 7 \le x < \infty \end{cases}$$

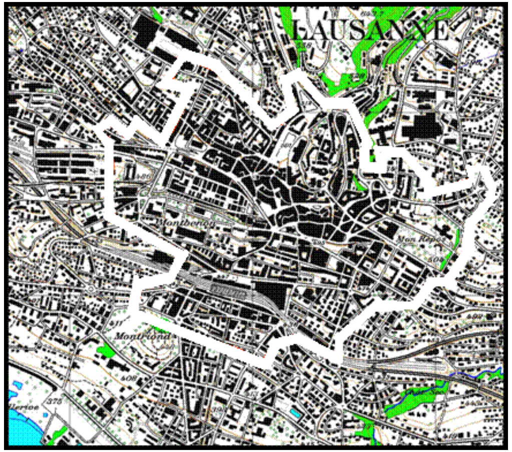


- $\hat{F}(x)$ is the proportion of observations smaller or equal to x.

Example: traffic signal plans

- Road network of Lausanne, Switzerland
- Empirical cdfs of the average travel times under different signal plans
- Goes beyond comparing averages
- Information on the variance (e.g. travel time reliability)





Ref: Dumont and Bert (2006). Simulation de l'agglomération Lausannoise SIMLO. Technical Report

Outline

1. Uncertainty in transportation
2. **Expected waiting time of a bus passenger problem**

Expected waiting time

- Every morning, you arrive at a bus stop and wait patiently for the bus to arrive.
- A bus is scheduled to arrive every 20 minutes.
- Why does it feel like you wait on average more than 10 minutes?

Expected waiting time phenomenon

- Expected waiting time for a randomly arriving prospective rider
- Consider a prospective rider on this bus line who arrives at the stop at a random time, independently of the bus schedule (or without knowledge of the schedule).
- Suppose buses arrive at the stop, on average, every $E[X]$ time units.
  - X is called the bus headway.
  - Let σ_X be the bus headway variance.
- What then is the expected waiting time of this prospective rider until the next bus arrives?
  - Hint: The answer is not $E[X]/2$. Why not?

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Inspection paradox

- This is an example of a problem involving “random incidence”, “inspection paradox”.
  - See Chap. 2 in R.C. Larson A. Odoni, Urban Operations Research (2007)
- The inspection paradox arises whenever the probability of observing a quantity is related to the quantity being observed.

Source: Allen Downey, The Inspection Paradox Is Everywhere (2015). Blog post.

References

1. Larson, Richard C. and Amedeo R. Odoni. **Urban Operations Research**. Prentice-Hall (1981). Chapter 2: Probability.

2. Lectures Modeling and Simulation in Logistics (2007) by Profs. Thomas Liebling and Carolina Osorio

3. Lecture Mesoscopic Simulation, Prof. Michel Bierlaire

4. Slides adapted from Carolina Osorio.

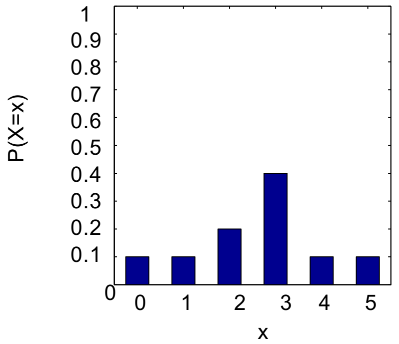
Appendix: Probability Review

Discrete random variable

- Random variable: outcome of random experiment
- Discrete random variable (r.v.) X, Y, etc.
- Realizations: x, y, etc.
- Defined over a set of discrete values: $\{0, \ldots, K\}$
- Is the outcome of a random experiment according to the probability distribution, called the probability mass function (pmf):

$$P(X = k) = f_k, \qquad k = 0, \ldots, K$$

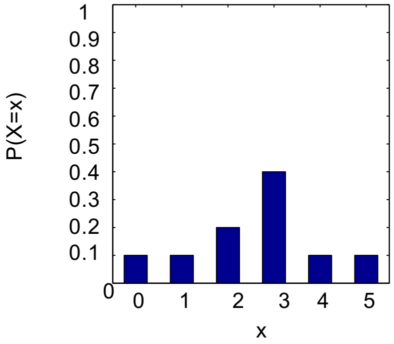
Such that $\begin{cases} f_k \geq 0, k = 0, \ldots, K \\ \sum_{k=0}^{K} f_k = 1 \end{cases}$

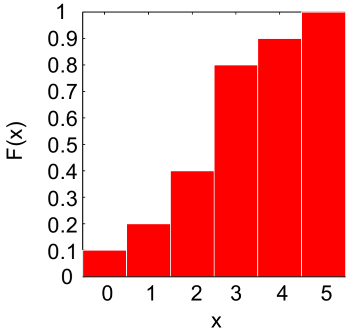


Cumulative distribution function (cdf): $F(x)$

$$F(x) = P(X \leq x) = \sum_{k \leq x} f_0 + \ldots + f_{\lfloor x \rfloor}$$

$F(x)$ is the probability that X is smaller or equal to x.





Example. For $x = 2, F(x) = f_0 + f_1 + f_2$

$F(x)$ is such that:

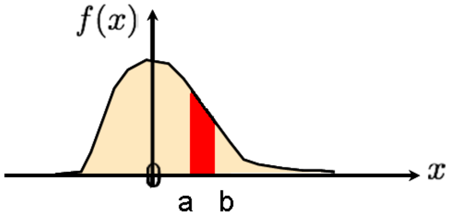
- Bounded: $0 \leq f(x) \leq 1$
- Monotone increasing: If $x \leq y$, then $F(x) \leq F(y)$

Expected value and variance

- Expectation:
  - $E[X] = \sum_{k=0}^{K} k f_k$
- Variance:
  - $Var[X] = E[(X - E[X])^2]$
    $= E[X^2] - 2E[X]^2 + E[X] = E[X^2] - E[X]^2$
  - Squared deviation of a random variable from its mean
  - For a discrete distribution with potential outcomes $k = 0, \dots, K$:
    - $Var[X] = \sum_{k=0}^{K} (k - E[X])^2 f_k$
      $= \sum_{k=0}^{K} k^2 f_k - E[X]^2$
- Standard deviation:
  - $\sigma = \sqrt{Var[X]}$

Continuous random variable

- Continuous random variable (r.v.) X, Y, etc.
- Realizations: x, y, etc.
- Probability density function: $f_X(x)$
- Defined by a continuous interval $[a, b]$, could be infinite



- E.g. $\int_{-\infty}^{\infty} f_X(x)dx = 1$
- Nonnegativity: $f_X(x) \geq 0, \forall x$
- Cumulative distribution function (cdf): $F_X(x)$

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u)du$$

Continuous random variable

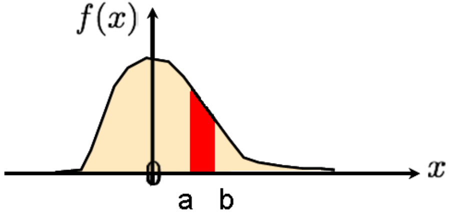
- Expectation:

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

- Variance:

$$\sigma_X^2 = Var(x) = \int_{-\infty}^{\infty} (x f(x) - E[X]^2) dx = E[X^2] - (E[X])^2$$

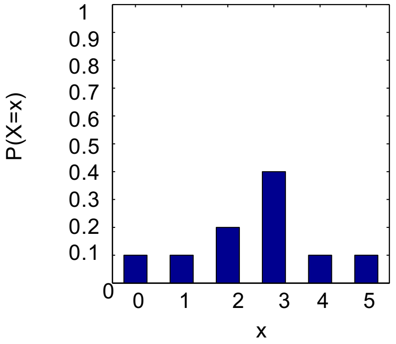
- $P(a \le X \le b) = \int_a^b f(x) dx$ is equal to the area:



Discrete vs. continuous distributions

Probability mass function:

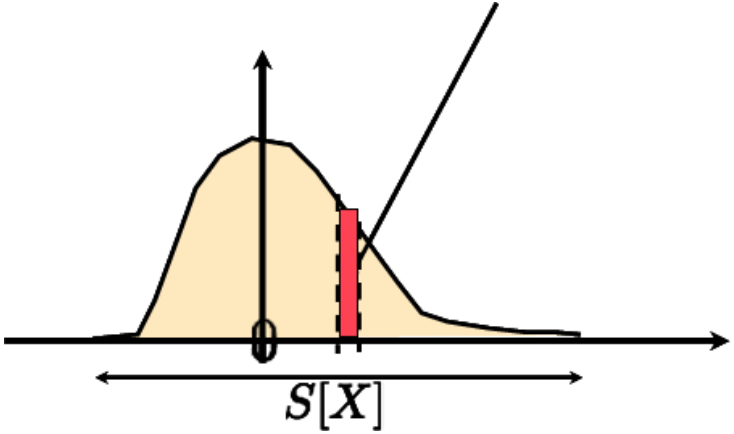
$$P(X - k) = f_k, \qquad k = 0, \ldots, K$$



Example: uniform distribution

Probability density function:

$$P(x \leq X \leq x + dx) = f(x)dx$$



Support: locations where f(x) is nonzero