

# Uncertainty in transportation

Probabilistic Concepts

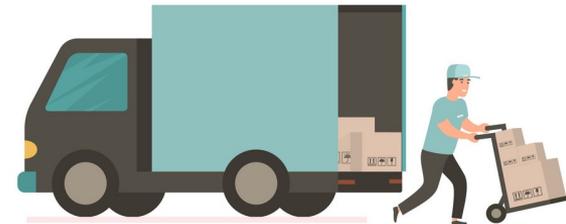
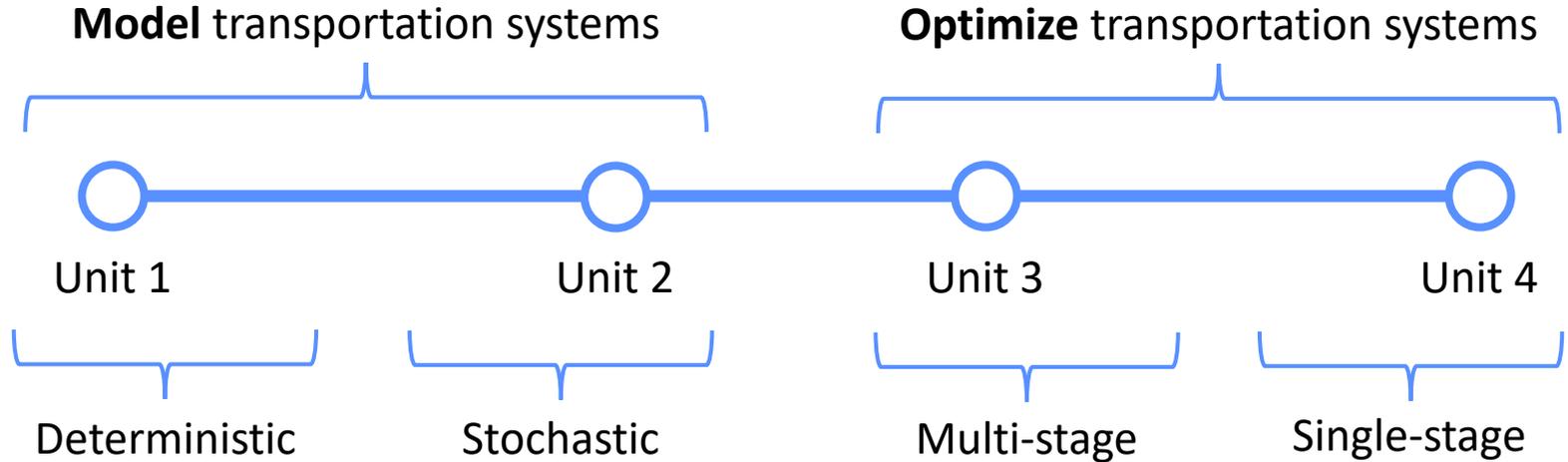
**Cathy Wu**

1.041/1.200 Transportation: Foundations and Methods

# Readings

1. Larson, Richard C. and Amedeo R. Odoni. **Urban Operations Research**. Prentice-Hall (1981). Chapter 2: Probability. [URL](#).

# 1.041/1.200 overview



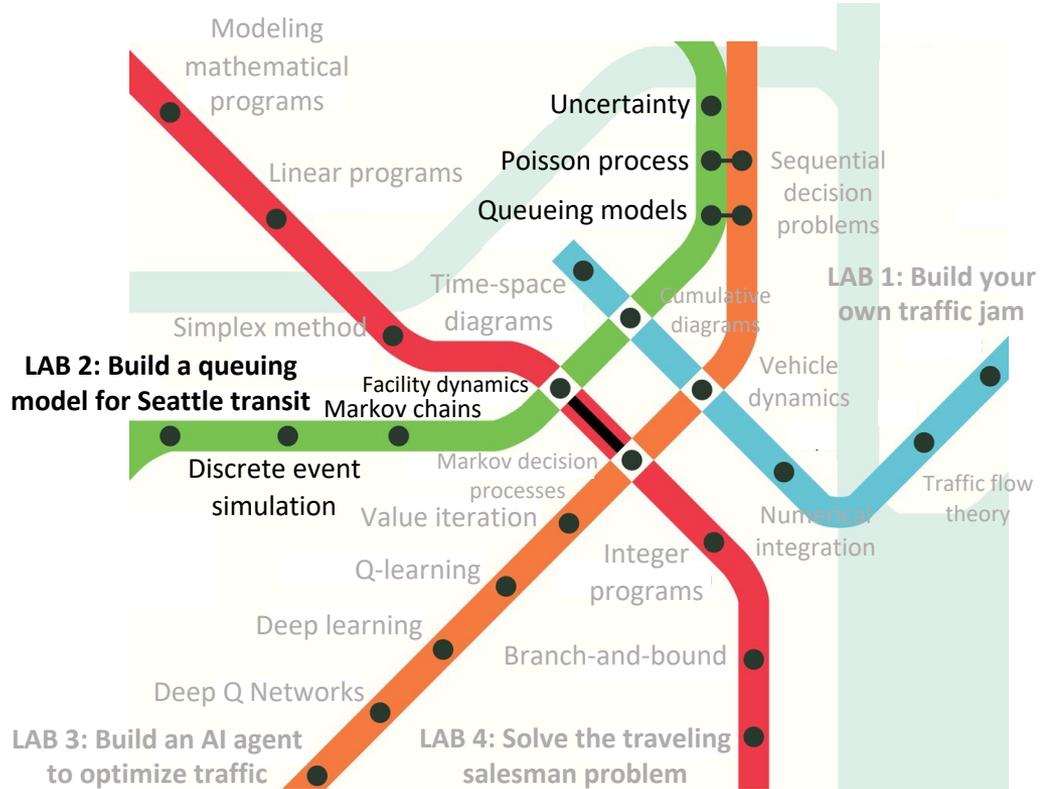
# Unit 2: Queuing systems



Unit 2

Modeling

Stochastic



# Outline

1. Uncertainty in transportation
2. Expected waiting time of a bus passenger problem

# Uncertainty

- Decision-making typically takes place under uncertainty
- Today: deterministic → probabilistic concepts.
- What is the role of probability for modeling in transportation?

# Outline

1. **Expected waiting time of a bus passenger problem**
2. Uncertainty in transportation

# Expected waiting time

- Every morning, you arrive at a bus stop and wait patiently for the bus to arrive.
- A bus is scheduled to arrive every 20 minutes.
- Why does it feel like you wait on average more than 10 minutes?



# Expected waiting time phenomenon

- Expected waiting time for a randomly arriving prospective rider
- Consider a prospective rider on this bus line who arrives at the stop at a random time, independently of the bus schedule (or without knowledge of the schedule).
- Suppose buses arrive at the stop, on average, every  $E[X]$  time units.
  - $X$  is called the bus headway.
  - Let  $\sigma_X$  be the bus headway variance.
- What then is the expected waiting time of this prospective rider until the next bus arrives?
  - Hint: The answer is not  $E[X]/2$ . Why not?

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# Inspection paradox

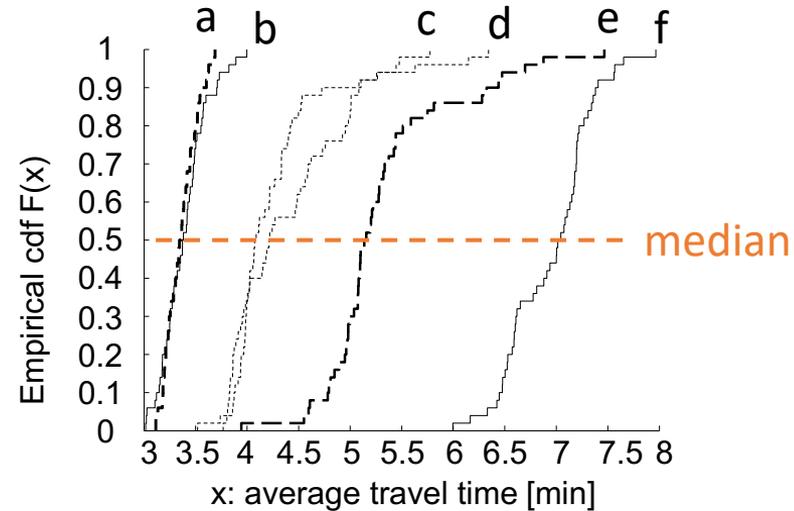
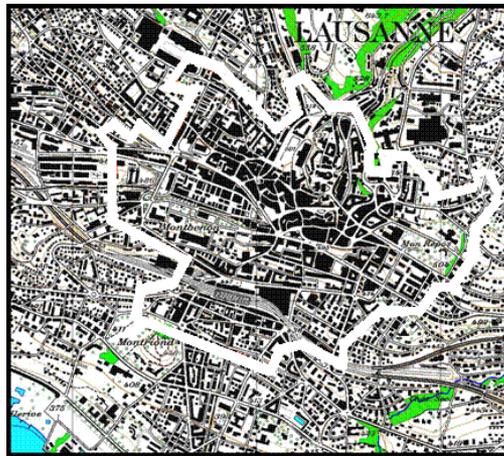
- This is an example of a problem involving “random incidence”, “inspection paradox”.
  - See Chap. 2 in R.C. Larson A. Odoni, Urban Operations Research (2007)
- The **inspection paradox** arises whenever the probability of observing a quantity is related to the quantity being observed.

# Outline

1. Expected waiting time of a bus passenger problem
2. **Uncertainty in transportation**

# Example: traffic signal plans

- Road network of Lausanne, Switzerland
- Empirical cdfs of the average travel times under **different signal plans**
- Goes beyond comparing averages
- Information on the variance (e.g. travel time reliability)



# Examples: Sources of uncertainty in transportation

- Road traffic
  - Weather, road condition
  - Local events
  - Time of day (demands)
  - Day of week (demands)
  - Types of drivers on the road
  - Exact number of vehicles, driving behavior
- Bus stop
  - Exact arrivals of passengers
  - Road traffic
- Uber/Lyft ride
  - Availability of drivers
  - Other users making requests
  - Road traffic
  - Exact traffic signal timing
- Airline departure
  - Engine malfunction, safety checks
  - Boarding time
  - Run-way availability

# References

1. Larson, Richard C. and Amedeo R. Odoni. **Urban Operations Research**. Prentice-Hall (1981). Chapter 2: Probability.
2. Lectures Modeling and Simulation in Logistics (2007) by Profs. Thomas Liebling and Carolina Osorio
3. Lecture Mesoscopic Simulation, Prof. Michel Bierlaire
4. Slides adapted from Carolina Osorio.

## *Appendix: Probability Review*

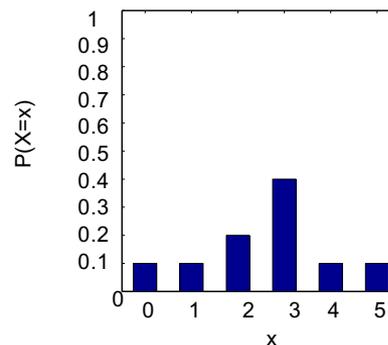
# Discrete random variable

- Random variable: outcome of random experiment
- Discrete random variable (r.v.)  $X, Y$ , etc.
- Realizations:  $x, y$ , etc.
- Defined over a set of discrete values:  $\{0, \dots, K\}$
- Is the outcome of a random experiment according to the probability distribution, called the **probability mass function** (pmf):

$$P(X = k) = f_k, \quad k = 0, \dots, K$$

Such that

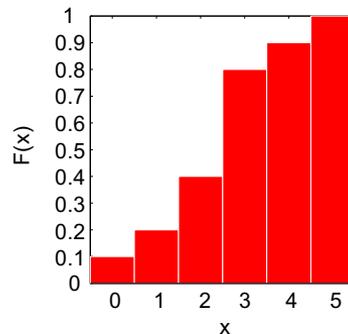
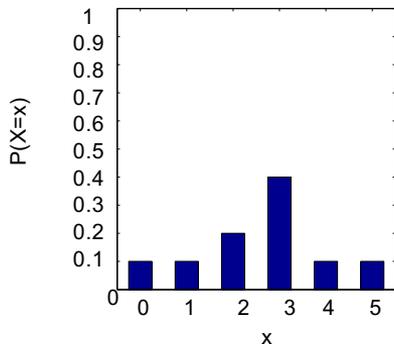
$$\begin{cases} f_k \geq 0, k = 0, \dots, K \\ \sum_{k=0}^K f_k = 1 \end{cases}$$



# Cumulative distribution function (cdf): $F(x)$

$$F(x) = P(X \leq x) = \sum_{k \leq x} f_0 + \dots + f_{\lfloor x \rfloor}$$

$F(x)$  is the probability that  $X$  is smaller or equal to  $x$ .



Example. For  $x = 2$ ,  $F(x) = f_0 + f_1 + f_2$

$F(x)$  is such that:

- **Bounded:**  $0 \leq f(x) \leq 1$
- **Monotone increasing:** If  $x \leq y$ , then  $F(x) \leq F(y)$

# Expected value and variance

- Expectation:

- $E[X] = \sum_{k=0}^K k f_k$

- Variance:

- $$\begin{aligned} \text{Var}[X] &= E[(X - E[X])^2] \\ &= E[X^2] - 2E[X]^2 + E[X] = E[X^2] - E[X]^2 \end{aligned}$$

- Squared deviation of a random variable from its mean

- For a discrete distribution with potential outcomes  $k = 0, \dots, K$ :

- $$\begin{aligned} \text{Var}[X] &= \sum_{k=0}^K (k - E[X])^2 f_k \\ &= \sum_{k=0}^K k^2 f_k - E[X]^2 \end{aligned}$$

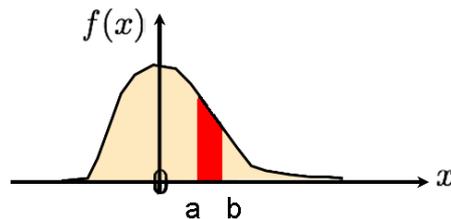
- Standard deviation:

- $\sigma = \sqrt{\text{Var}[X]}$

# Continuous random variable

- Continuous random variable (r.v.)  $X, Y$ , etc.
- Realizations:  $x, y$ , etc.
- Probability density function:  $f_X(x)$
- Defined by a continuous interval  $[a, b]$ , could be infinite

- E.g.  $\int_{-\infty}^{\infty} f_X(x) dx = 1$



- Nonnegativity:  $f_X(x) \geq 0, \forall x$
- Cumulative distribution function (cdf):  $F_X(x)$

$$F(x) = P(X \leq x) = \int_{-\infty}^x f(u) du$$

# Continuous random variable

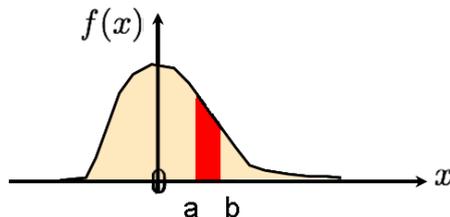
- Expectation:

$$E[X] = \int_{-\infty}^{\infty} xf(x)dx$$

- Variance:

$$\sigma_X^2 = Var(x) = \int_{-\infty}^{\infty} (xf(x) - E[X])^2 dx = E[X^2] - (E[X])^2$$

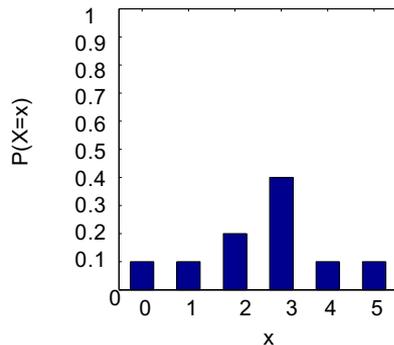
- $P(a \leq X \leq b) = \int_a^b f(x)dx$  is equal to the area:



# Discrete vs. continuous distributions

Probability mass function:

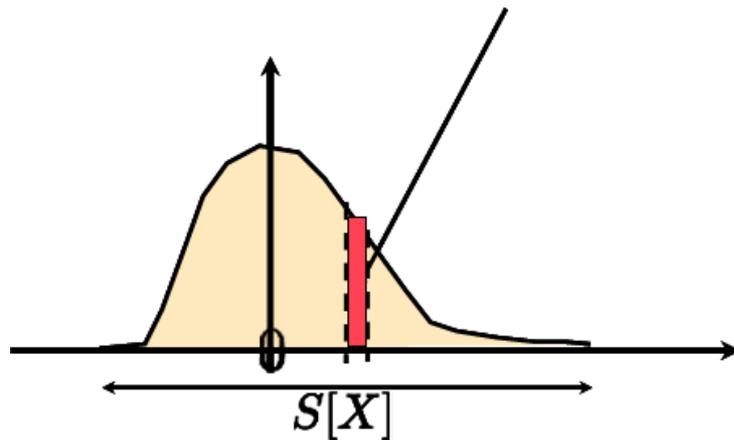
$$P(X = k) = f_k, \quad k = 0, \dots, K$$



Example: uniform distribution

Probability density function:

$$P(x \leq X \leq x + dx) = f(x)dx$$



Support: locations where  $f(x)$  is nonzero

# Empirical (cumulative) distribution function

- From relative frequencies:
  - Example:  $k$  = time headway (minutes) of bus arrivals
- Compute cumulative frequencies:
- Yielding **empirical cumulative distribution function (cdf)**:

$$\hat{F}(x) \begin{cases} 0 & : -\infty < x < 2 \\ c_k & : k \leq x < k + 1, k = 2, \dots, 6 \\ 1 & : 7 \leq x < \infty \end{cases}$$

- $\hat{F}(x)$  is the proportion of observations smaller or equal to  $x$ .

$k$	2	3	4	5	6	7
$v_k = \frac{n_k}{n}$	.1	.1	.2	.4	.1	.1
$k$	2	3	4	5	6	7
$c_k$	.1	.2	.4	.8	.9	1

