Queueing theory fundamentals

Stochastic Delays

Cathy Wu

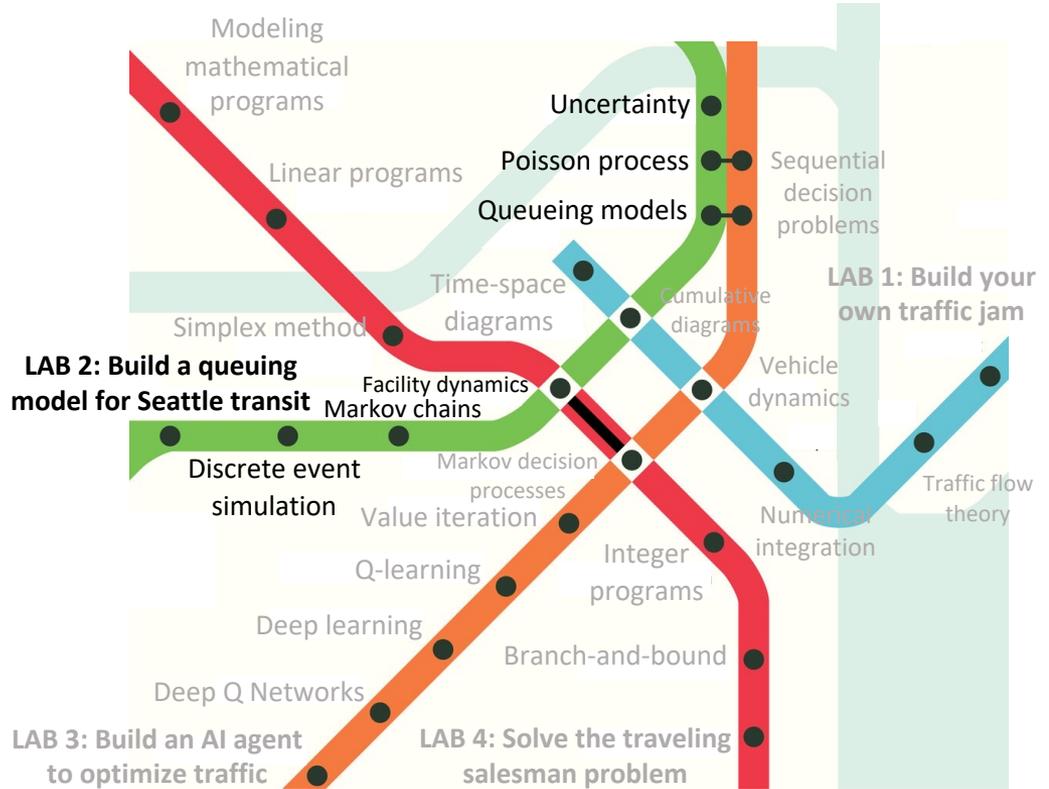1.041/1.200 Transportation: Foundations and Methods

Readings

1. Larson, Richard C. and Amedeo R. Odoni. **Urban Operations Research**. Prentice-Hall (1981). Chapter 2: Probability. URL.

Unit 2: Queuing systems

Unit 2

Modeling

Stochastic

Modeling mathematical programs

Linear programs

Uncertainty

Poisson process

Queueing models

Sequential decision problems

Time-space diagrams

Cumulative diagrams

LAB 1: Build your own traffic jam

Simplex method

LAB 2: Build a queuing model for Seattle transit

Facility dynamics
Markov chains

Vehicle dynamics

Discrete event simulation

Markov decision processes

Value iteration

Numerical integration

Traffic flow theory

Q-learning

Integer programs

Deep learning

Branch-and-bound

Deep Q Networks

LAB 3: Build an AI agent to optimize traffic

LAB 4: Solve the traveling salesman problem

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Outline

1. Delays in transportation

2. Queueing theory fundamentals

3. Problem: timing of a pedestrian crossing light

Outline

1. **Delays in transportation**

2. Queueing theory fundamentals

3. Problem: timing of a pedestrian crossing light

Delays in transportation

- Delays are pervasive in transportation

- Today: Introduce stochastic models of delay
  - To diagnose and improve delay
  - Framework: Queueing theory

Local delays

#	U.S. city	Hours lost per year per driver
1	Boston	164
2	Washington, D.C.	155
3	Chicago	138
4	Seattle	138
5	New York City	133
6	Los Angeles	128

Sources:
Boston.com
Boston Magazine
[February, 2019]

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TRANSPORTATION

Boston Has the Worst Traffic in the Country

Drivers in Boston are spending more time sitting in their cars than people in Los Angeles.

by ELLEN GERST • *2/12/2019, 10:06 a.m.*

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Delays in transportation

- Congestion/delays arises across all transportation modes

- Urban congestion:
  - "Urban congestion lead in 2007 to an estimated additional 4.2 billion hours of travel and 2.8 billion gallons of fuel with a cost of $87.2 billion across urban areas in the US, an increase of more than 50% over the previous decade"

- Highway traffic:
  - "Highway vehicle miles traveled are projected to grow 60%, from 2,952 billion miles traveled in 2005 to 4,733 billion miles traveled in 2030."

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Delays in transportation

- **Air traffic:**
  - "The airline industry's on-time performance in the first seven months of 2007 was the worst on record, and nationally almost 30% of all flights are now cancelled or substantially delayed."
  - "Aircraft travel is projected to nearly double, and current forecasts estimate over 1.5 billion air passengers annually by 2030. This will place unparalleled demand on the air system."

- **Freight traffic:**
  - " The U.S. transportation system currently moves over 50 million tons of freight on the US transportation network. (...) By 2035, tons transported overall are expected to double to over 100 million, placing incomparable pressure on our domestic transportation network."

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Outline

1. Delays in transportation

2. **Queueing theory fundamentals**
   a. **Exponential distribution**
   b. Poisson process

3. Problem: timing of a pedestrian crossing light

Background for queuing theory

- Modeling: **directional** correctness vs **absolute** correctness

- To be useful, the assumed distributional form, should be:
    1. Sufficiently realistic (reasonable predictions)
    2. Sufficiently simple (mathematically tractable)

- Queueing theory is a key modeling framework for transportation systems

- Key tools
    - The Poisson process
    - The exponential distribution, which is intimately tied to the Poisson process

Exponential distribution

- Also called the negative exponential

- $T \sim \mathrm{Exp}(\lambda)$

- λ is a parameter (a constant)

- PDF: $f_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$

- $E[T] = \dfrac{1}{\lambda}$

- $Var[T] = \dfrac{1}{\lambda^2}$

- CDF: The probability that T will be less than or equal to any particular constant value, t, is equal to:

$$F_T(t) = P(T \leq t) = \int_{x=0}^{t} \lambda e^{-\lambda x} dx = 1 - e^{-\lambda t}$$
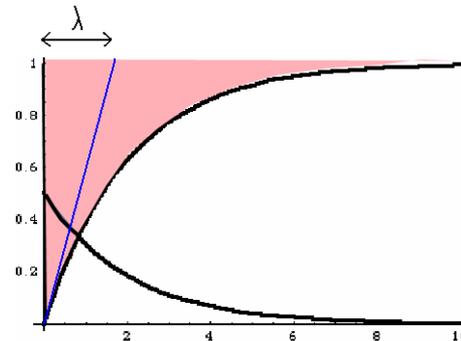
# Exponential distribution: numerical example

Gap acceptance problem

- Suppose that T:
    a) represents the times between successive passages (headways) of vehicles on a road; and
    b) is described by an exponential pdf.

- A driver requires 5 sec to insert into mainstream traffic that flows at 400 veh/hr

- What is the probability that the next gap is acceptable?

Exponential dbn

- What are the implications of assuming $T \sim Exp(\lambda)$?

Main properties of the exponential dbn:

- Prop. 1) $f_T(t)$, the pdf, is a strictly decreasing function of t $(t \geq 0)$
  - $P(0 \leq T \leq \Delta t) > P(t \leq T \leq t + \Delta t), \; t > 0, \; \Delta t > 0$
  - $P\left(0 < T < \frac{E[T]}{2}\right) = 0.393$
  - $P\left(\frac{E[T]}{2} < T < \frac{3E[T]}{2}\right) = 0.38$
  - So, T is more likely to be $< \frac{E[T]}{2}$ (small), than near $E[T]$, even though 2nd interval is twice as wide as the 1st.

- Implications in practice?
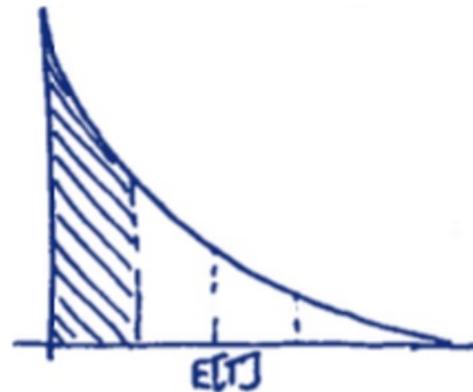
Exponential dbn

- Prop. 2) Lack of memory ("memoryless")
$$P(T > t + s | T \geq s) = P(T > t) \quad t > 0, \qquad s > 0$$

- Interpretation: The probability dbn of the remaining time until the occurrence of the next event (e.g. next vehicle arrival, next service completion) is always the same, regardless of how much time has already passed.

  - For inter-arrival times: the time until the next arrival is uninfluenced by when the last arrival occurred

  - For service times: if they differ from customer to customer, then "memoryless" may be a desirable property

- The exponential dbn is the only continuous dbn with this property.

- Also, a small interval is independent of the time:
$$P(T \leq t + \varepsilon | T > t) = \lambda \varepsilon + o(\varepsilon), \qquad \epsilon \ll 1$$

- Implications in practice?

Example: Jitney rider

Carla waits on the side of the road for a jitney*, which will transport her to the next town. This jitney travels along a fixed route between the edge of two towns.

An Mikrotrans angkot operated by Transjakarta in Jakarta

1. Suppose that the interarrival time of the jitney service is an exponentially distributed random variable with a mean of 10 minutes. Carla has already waited 15.5 minutes. What is the expected additional time (conditional mean) that she will have to wait?

2. Now assume the jitney interarrival time is uniformly distributed between 2 to 18 minutes. What is the expected additional time (conditional mean) that Carla will have to wait?

*A jitney is a form of unlicensed taxi, often unscheduled. Jitney service (in various forms) is prevalent in developing countries.
Fun fact: Uber is technically an illegal taxi service in jurisdictions with medallion systems that restrict the number of legal cabs in operation.

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Exponential dbn

- Prop. 3) Minimum of Exp. R.V.s yields an Exp. R.V.

$$T_i \sim Exp(\lambda_i), \qquad \forall i \in 1, 2, \ldots, n \quad \text{independent}$$

$$U = \min_{i=1,\ldots,n} T_i$$

$$U \sim Exp\left(\sum_{i=1}^{n} \lambda_i\right)$$

- Inter-arrivals of different types, then the time until the next arrival follows an exponential dbn.

- Example: n check-in counters currently attending customers, then the time until the next service completion follows an exponential dbn.

Exponential dbn

- Prop. 4) Relationship to the Erlang distribution.

- k independent r.v's: $T_i \sim Exp(\lambda), \ \forall i \in 1, 2, \dots, k$
  Then, $E_k = \sum_{i=1}^{k} T_i$ follows a kth-order Erlang dbn.

$$f_{E_k}(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}, \qquad x \geq 0, \qquad k = 1, 2, \dots$$

- $E[E_k] = \frac{k}{\lambda}$

- $Var(E_k) = \frac{k}{\lambda^2}$

- Example: **total** waiting time of passengers for a bus

- Note (sanity check): the exponential dbn coincides with $k = 1$.

Outline

1. Delays in transportation

2. **Queueing theory fundamentals**
   a. Exponential distribution
   b. **Poisson process**

3. Problem: timing of a pedestrian crossing light

The Poisson process

- A **counting process**: Counts the number of events $N(t)$ that occur during time interval $[0, t]$.

- If successive inter-event (e.g., inter-arrival) times are mutually independent and identically distributed as $Exp(\lambda)$, then: $\{N(t), t \geq 0\}$ is a **Poisson process** with rate λ.

- A random variable X follows a **Poisson distribution** if

  - Probability mass function: $f(k; \lambda) := \frac{\lambda^n e^{-\lambda}}{k!}$

  - $E[X] = Var(X) = \lambda$

- The number of events in an interval of length t is a Poisson r.v. with parameter λt

  - $P(N(t) = n) = f(n; \lambda t) = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$

    - Probability that n events take place during the time interval
    - t: time interval length
    - λ: arrival rate (veh/unit time), also referred to as the "intensity" of the arrivals of Poisson events

  - $E[N(t)] = Var\big(N(T)\big) = \lambda t$

Probability mass function
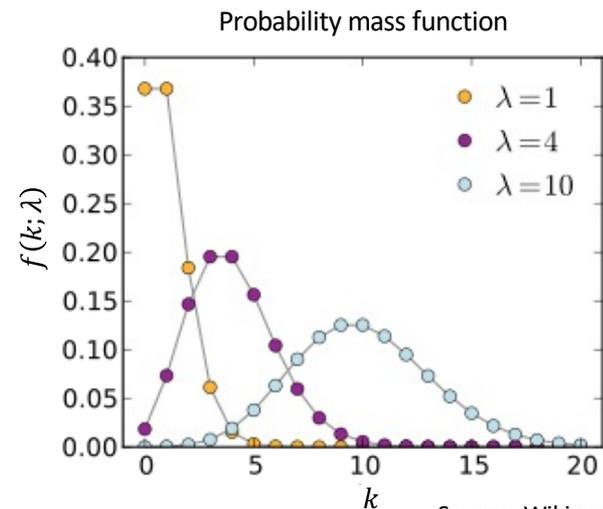
Source: Wikipedia

Poisson process ➔ Exponential dbn

- Denote T as the time between successive arrivals, then during a time interval of length $t \leq T$, no arrivals occurred, so for all such t:

$$P(t \leq T) = P(N(t) = 0) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$$

- Cumulative dbn function:

$$F(t; \lambda) = P(0 \leq T \leq t) = 1 - e^{-\lambda t}$$

- Probability density function:

$$f(t; \lambda) = \frac{dF(t; \lambda)}{dt} = \lambda e^{-\lambda t}$$

- Inter-arrival times follow an exponential dbn, with parameter λ.

Inter-arrival times that are i.i.d. $Exp(\lambda)$ is equivalent to arrivals according to a Poisson process $P(\lambda)$

Fundamental properties of the Poisson process

1. For a sufficiently small Δt:
    $$P(N(t + \Delta t) - N(t) = 1) \simeq \lambda \Delta t$$
    $$P(N(t + \Delta t) - N(t) = 0) \simeq 1 - \lambda \Delta t$$
    $$P(N(t + \Delta t) - N(t) \geq 2) \simeq 0$$
    where these probabilities represent, respectively: the probability that exactly one event, no event, or two or more events will occur in the next Δt

2. "Memoryless"
    - The number of events that occur during any pre-specified interval of length Δt does not depend on the "starting time" of the time interval or on the number of events recorded prior to the time interval
    - If we begin observing a Poisson process with rate λ at $t = 0$, the pdf for the time X until the next arrival is given by $f_X(t) = \lambda e^{-\lambda t}, t \geq 0$ no matter how long before $t = 0$ the last arrival occurred.

3. The number of events that occur in disjoint time intervals are mutually independent r.v.'s

4. The sum of K independent Poisson processes is a Poisson process with a rate equal to the sum of the K rates.

Poisson events and Bernoulli trials

1. Given two independent Poisson processes, with rates λ_1 and λ_2, respectively. Let X_1 and X_2 be the time until the next arrival from each process.

   - Then, the probability of an arrival from Process 1 before Process 2 is:

$$P(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2} := p$$

   - And the probability that the earlier arrival will be from Process 2 is:

$$q = 1 - p = \frac{\lambda_2}{\lambda_1 + \lambda_2}$$

2. Thus, the type of arrival (from Process 1 or from Process 2) is determined by a Bernoulli trial ("coin flip").

   - The probability that, out of n arrivals, m will be from Process 1 is given by:
$$P(m \text{ successes in } n \text{ trials}) = C_m^n p^m (1 - p)^{n-m}, \qquad 0 \le m \le n$$

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Example: Timing of a pedestrian crossing light*

- Pedestrians approach from the left (resp. right) side of the crossing in a Poisson manner with arrival rate λ_L (resp. λ_R) arrivals per minute. Pedestrians wait until light turns green, referred to as a "dump". Assume:
  - all pedestrians cross instantaneously
  - the dump duration is zero
  - left and right arrival processes are independent.

- We consider three operating rules:
  - A. Dump every t_1 minutes
  - B. Dump whenever the <u>total</u> number of waiting pedestrians equals n_0
  - C. Dump when the first pedestrian to arrive, after the previous dump, has waited for t_2 minutes

- For each rule, determine:
  1. The expected number of pedestrians crossing left to right on any dump
  2. The probability that zero pedestrians cross left to right on any particular dump
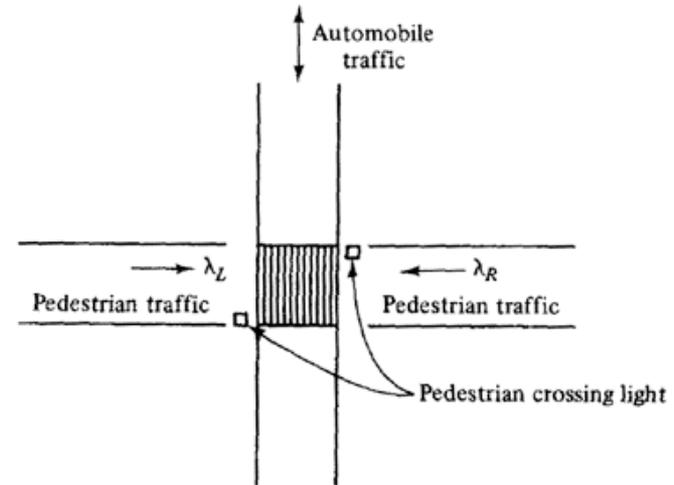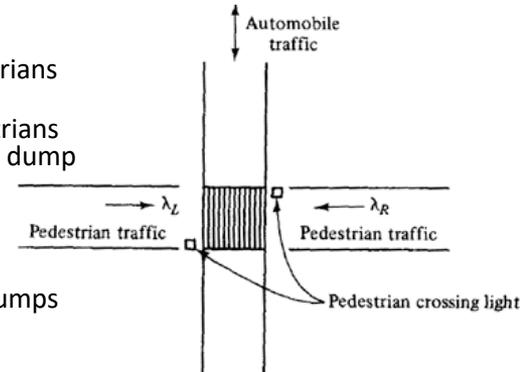 3. The pdf for the time between dumps

FIGURE 2.14 Pedestrian crossing problem.

Rule C: Dump whenever the *first* pedestrian to arrive after the previous dump has waited T_0 minutes.

Example: Timing of a pedestrian crossing light*

- Arrival rates λ_L, λ_R

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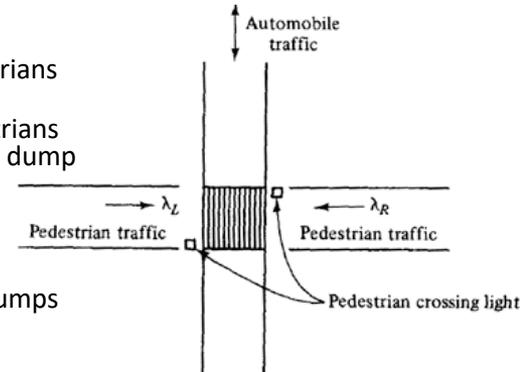
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Source: Chapter 2 in Larson, R.C. and A. Odoni, Urban Operations Research, Dynamic Ideas, Belmont, MA 2007

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References

1. Larson, Richard C. and Amedeo R. Odoni. **Urban Operations Research**. Prentice-Hall (1981). Chapter 2: Probability.

2. Walrand, Jean. Probability in Electrical Engineering and Computer Science: An Application-Driven Course. Quorum Books (2014).

3. Slides adapted from Carolina Osorio