Markov chains

Beyond basic facility dynamics

Cathy Wu

1.041/1.200 Transportation: Foundations and Methods

Readings

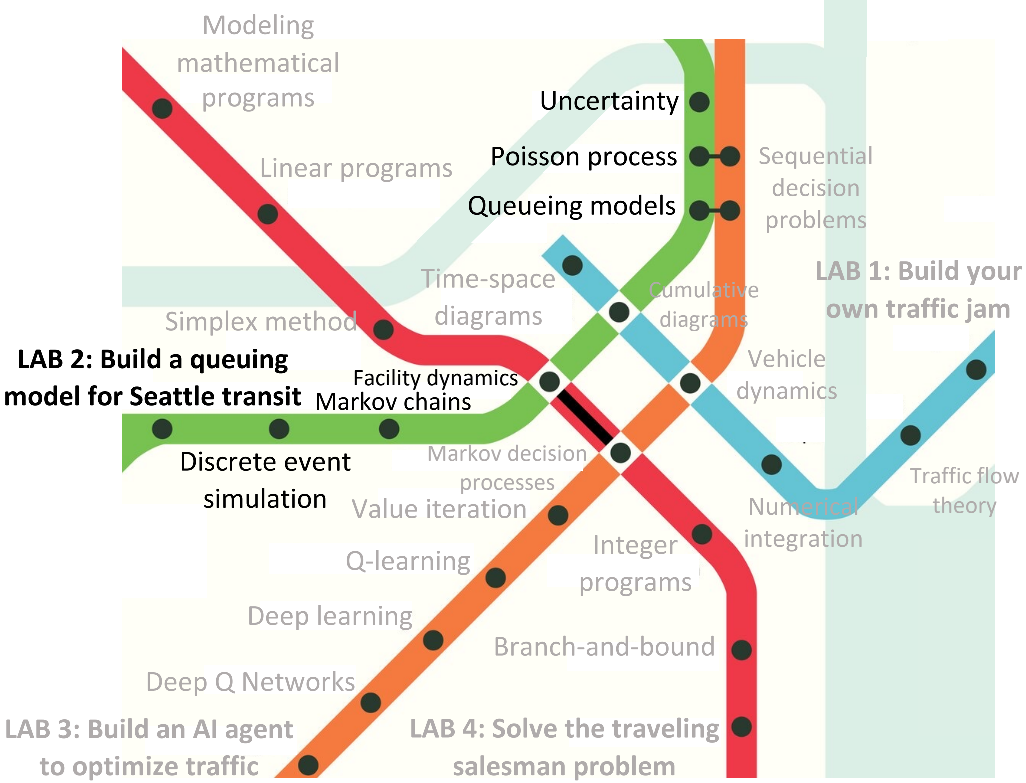
1. (Optional) Prof Ayalvadi Ganesh. "Markov Chains." Lecture notes. URL.

Unit 2: Queuing systems

Unit 2

Modeling

Stochastic



Outline

1. Transition rate matrix
2. Markov chains

Outline

1. **Transition rate matrix**
   a. Global balance equations
   b. Container terminal problem
2. Markov chains

Beyond the basic queues

- In this lecture, we discuss a more general strategy for modeling and analyzing more sophisticated facility dynamics.
  - We will introduce ideas of: transition rate matrix, global balance equations. We will leverage fundamental results from Markov chains.
- Solutions will no longer be analytical (in general) and can be instead computed numerically.
  - For now, we will still use properties of the Exponential distribution.
  - In Lecture 10, we will relax this to allow modeling of (even) more general facility dynamics (see Computational Lab #2).
- Why do we need fancier tools?

General strategy for stationary analysis

- Time perspective (what we've considered so far):
  - The random process $X_t \in \mathbb{N}$ evolves according to transition rates.
  - Represented by state transition diagram and/or transition rate matrix Q. ?
- Distributional perspective: $P_n(t) := P(X_t = n)$ is the proportion of time the system spends in state n.
  - Discrete-time form: $P_n(k) := P(N(\Delta t\ k) = n)$
  - $P(k) := \left(P_n(k)\right)_{n \in \mathbb{N}}$

1. Define the evolution of $P(k)$ as: ?

$$P(k+1)^T = P(k)^T T$$

   - Where $T := I + Q\Delta t$ is the transition probability matrix
2. Invoke fundamental theorem of Markov Chains. ?
   - Gives convergence to stationary distribution: $P(k) \to P$

Stationary analysis: in a nutshell

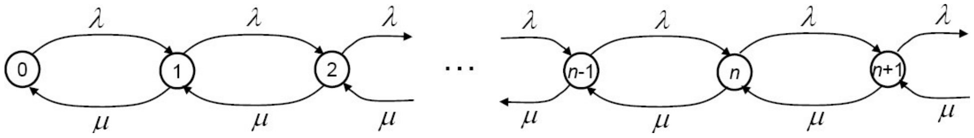
- Stationary distribution: often denoted P or π
- In applications:
  1. Define the state space
  2. Define the state transition diagram or the **transition rate matrix**
  3. Then, derive the balance equations to obtain the stationary distribution
  4. Derive $\overline{N}, \overline{N}_q, \overline{T}, \overline{T}_q$

Transition rate matrix

- Define transition rate matrix $Q = (q_{ij})_{i \in I, j \in J}, = \begin{bmatrix} q_{00} & q_{01} & \cdots \\ q_{10} & q_{11} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$
  - where q_{ij} is transition rate from state i to state j
  - $q_{ii} = -\sum_{j \neq i} q_{ij}$: rate of departure from state i
  - Square matrix
  - Helps describes the rate by which the system changes state, i.e., $\frac{d}{dt} P_n(t)$
- Possible transitions from a given state i, with their corresponding rates and conditions under which these transitions can take place

Example: M/M/1

initial state s	new state j	rate q_{ij}	condition
n	$n+1$	λ	always possible
n	$n-1$	μ	$n \geq 1$



Global balance equations

- The set of equations represented by the state transition diagram are called the global balance equations. They can be written as:

$$\sum_{j\neq i} P_j q_{ji} = -P_i q_{ii} \quad \forall i \quad \underset{\text{(equivalent)}}{\equiv} \quad P^T Q = 0$$

- In other words, in steady state (when $P(t) = P$), transitions into and out of states must be balanced.
- In some cases (e.g., M/M/c), the global balance equations can be directly solved for P (stationary analysis).
- In other cases…

Container terminal

- Arrival
  - Ships arrive at a container terminal according to a Poisson process with rate λ. Each ship brings thousands of containers.
  - There are c berths, i.e. only c ships can dock or anchor at the port simultaneously.
  - If all berths are occupied arriving ships must wait at the entrance of the port. A maximum of r ships can wait for entrance, further ships are redirected to another port.
- Berth operations
  - Upon arrival to the berth, a ship unloads its containers. The unloading time follows an exponential distribution with parameter μ.

- Departure
  - Once all containers are unloaded, the ship can leave the port, as long as the exit is clear, which occurs with probability p_f.
  - Suppose each berth has its own exit. If the exit is occupied, it remains so during an exponentially distributed time with parameter β.

Container terminal

Upon arrival to the port a ship (arrival rate λ):

1. [queue]
2. is served (service rate μ)
3. [blocked] (rate at which exit is occupied β, also exponentially distributed)
4. Departs (with probability p_f)

Note: Blocking after service: blocking mechanism used to describe how congestion arises and propagates

State space of port :

$$S = \{(A, B, W) \in N^3, A + B \leq c, W \leq r\}$$

- A = ships being served
- B = ships waiting to depart
- W = ships waiting to be served

Stationary distribution:

$$P = \big(P\big((A, B, W) = (a, b, w)\big), (a, b, w) \in S\big)$$

Problem: For $c = 2, r = 2$, define the possible transitions and their corresponding rates (tabulate).

Reminder (Departure): Once all containers are unloaded, the ship can leave the port, as long as the exit is clear, which occurs with probability p_f. Suppose each berth has its own exit. If the exit is occupied, it remains so during an exponentially distributed time with parameter β.

Container terminal

Outline

1. Transition rate matrix
2. **Markov chains**
    a. Python demo
    b. Application: (Simple) multi-lane modeling

Markov chain

- The sequence $\{X_t, t \geq 0\}$ that goes from state i to j with probability τ_{ij}, independently of the states visited before, is a Markov chain.
- τ_{ij} is also called a transition probability.
- Markov property: the current state contains all information for predicting the future of the process/chain.

Preview (Unit 4): Markov decision processes (MDP)

- Extension of Markov chains, where, in addition to the current state, the transition is also affected by **decisions (control)**, i.e. τ_{ijk}.
- Example of sequential decisions in transportation: traffic signal phase change, autonomous vehicle steering & acceleration, dynamic speed limits, rideshare matching

Transition probability matrix

- $T = \left(\tau_{ij}\right)_{i \in I, j \in J}$
- How come $P(k+1)^T = P(k)^T T$, where $T := I + Q\Delta t$?

Fundamental theorem of Markov chains

Distribution after k steps (each Δt, i.e. $t = \Delta t\ k$)

$$P_n(k+1) = \sum_{i \in N} P_i(k) \tau_{in}$$

Stationary distribution

$$P_n = \sum_{i \in N} P_i \tau_{in}$$

Often written in other texts (different notation) as: $\pi P = \pi$.

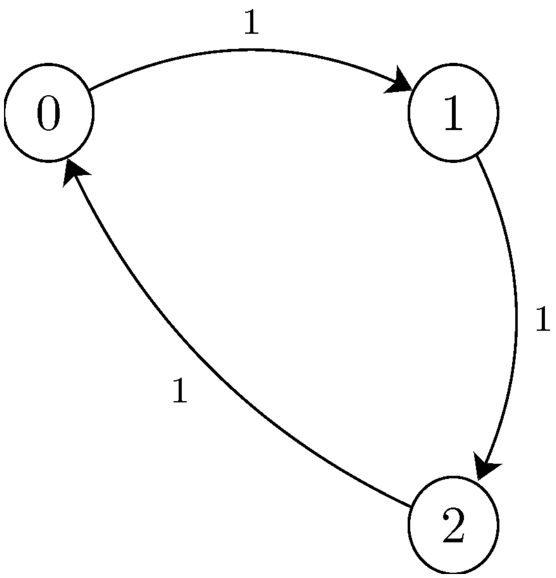
Definitions:

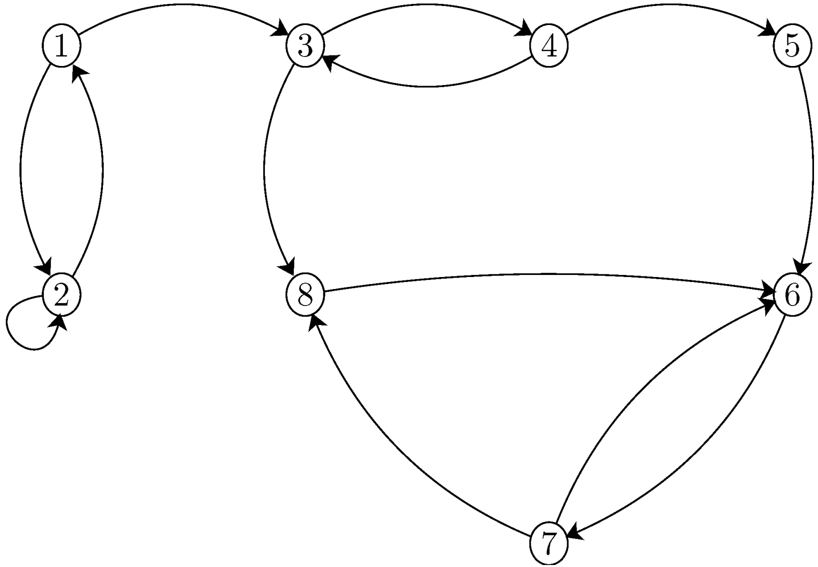
- Finite: finite number of possible states ($N < \infty$)
- Irreducible: can reach any state from any state, possibly after many steps
- Aperiodic: no state transitions back to itself **only** under cycle lengths of integer multiples > 1

Theorem (Fundamental theorem of Markov chains)

1. If the Markov chain is finite and irreducible, it has a unique invariant distribution P and P_n is the long-term fraction that $X(t) = n$.
2. If the Markov chain is also aperiodic, then the distribution $P(k)$ of $X(t)$ converges to P.

Example: Markov chains





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   a. **Python demo**
   b. Application: (Simple) multi-lane modeling

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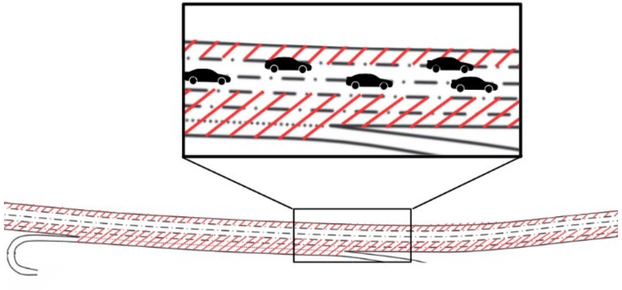
PICC

(Simple) multi-lane modeling

Goal: Enable analysis of vehicle controllers for more complex traffic, in particular for multi-lane roads.

Key challenges:

1. Driver behavior models too complex.
2. Steady-state analysis of car following models is less useful amidst frequent perturbations.



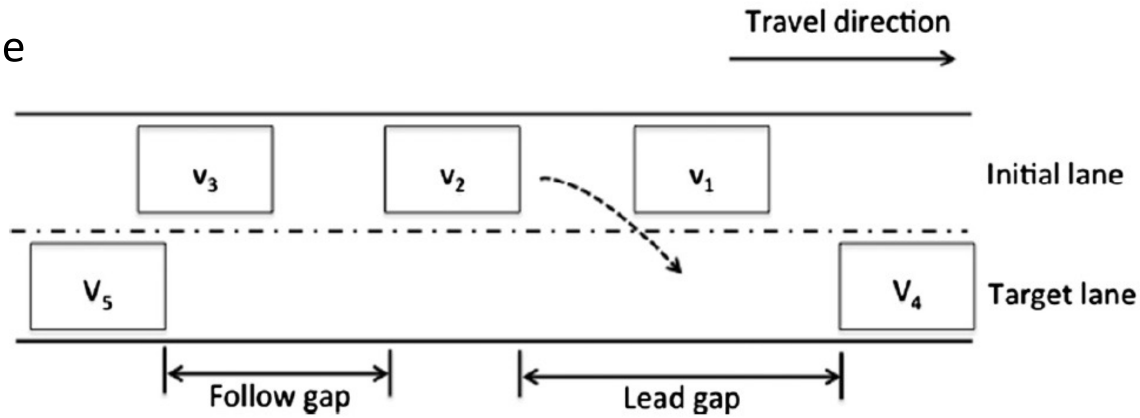
Contributions:

- *Multi-lane roads*: Reduction of a multi-lane model into a stochastic single-lane model.
- *Multi-lane counts*: Markov chain describing macroscopic multi-lane traffic, derived from microscopic vehicle dynamics.
- Enables analysis of variance of velocity, for understanding energy use.

Wu, Cathy, Eugene Vinitsky, Aboudy Kreidieh, and Alexandre Bayen. "Multi-lane reduction: A stochastic single-lane model for lane changing." IEEE ITSC, 2017.

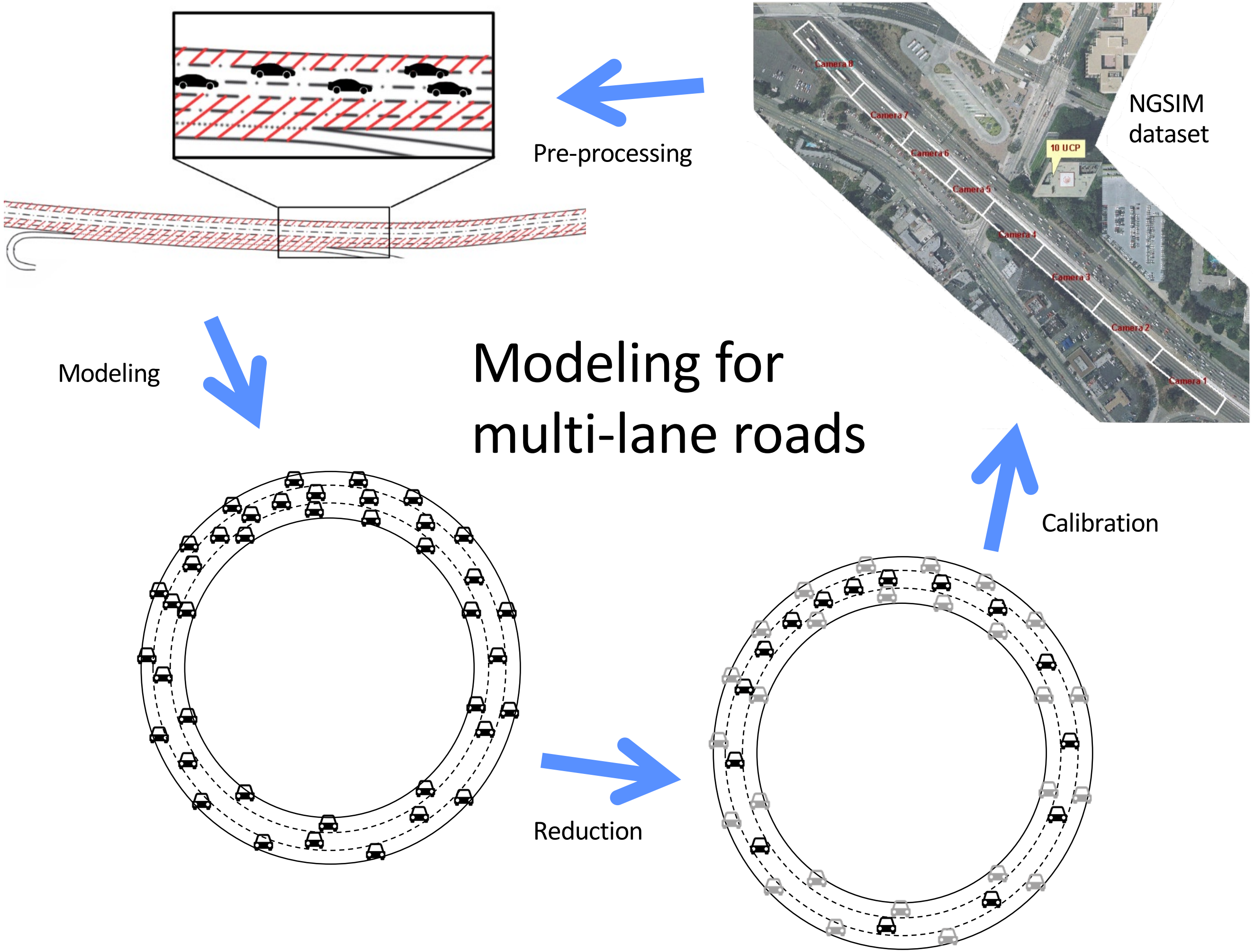
Dynamical System Lane Changing

- Major source of instability
- Focus on discretionary changes
- Multifactorial
  - Gap size
  - Lead and lag speed
  - Speed of adjacent lane
  - Braking distance
  - Cooperation
  - Driver politeness
  - Traffic density



Recent developments and research needs in modeling lane changing, Zuduo Zheng

Modeling for multi-lane roads



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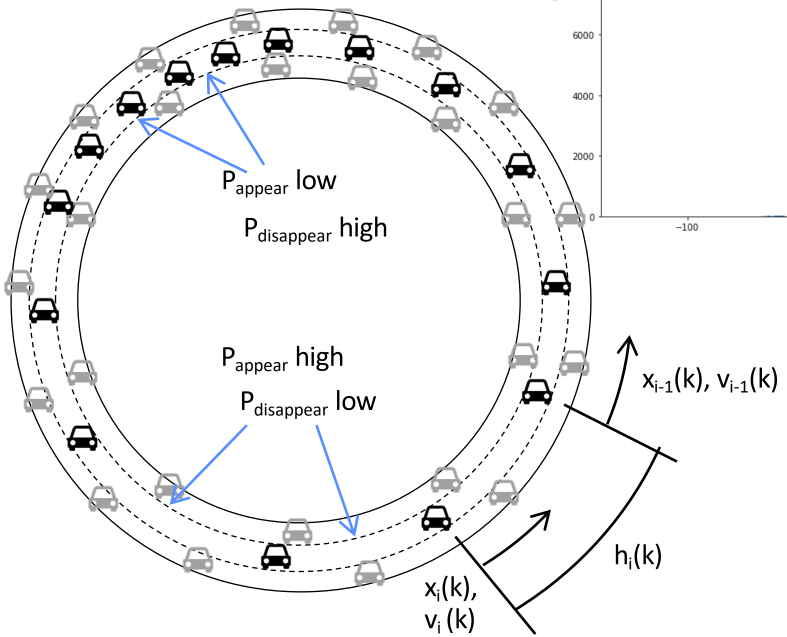
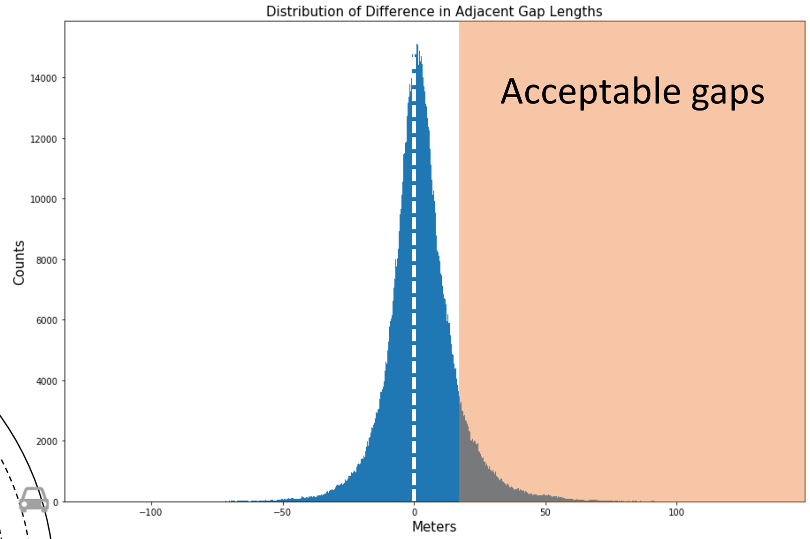


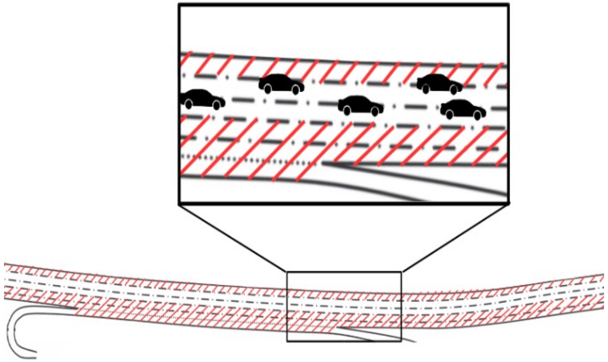


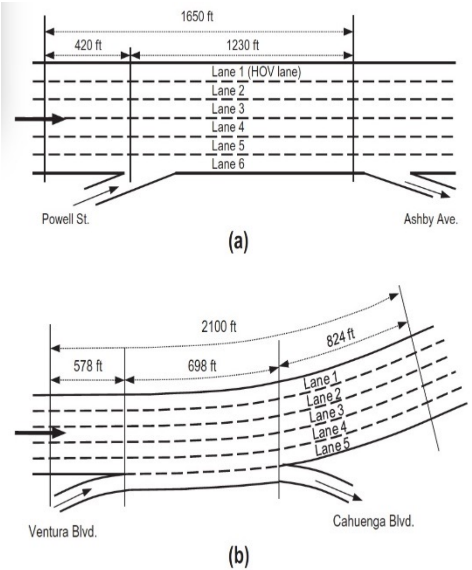
This distribution of differences in adjacent gap lengths from the NGSIM dataset indicates that there is plenty of probability mass with positive gap differences (the right side), providing opportunity for lane changes into the lane.

Wu, Cathy, Eugene Vinitsky, Aboudy Kreidieh, and Alexandre Bayen. "Multi-lane reduction: A stochastic single-lane model for lane changing." IEEE ITSC, 2017.

Model Calibration - NGSIM

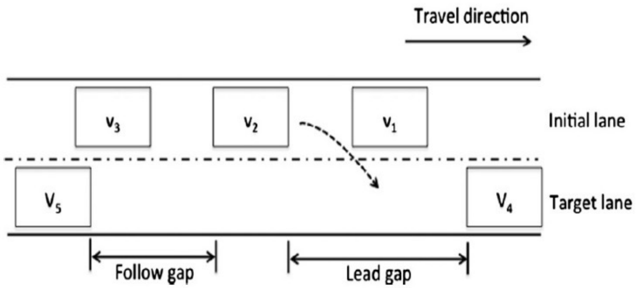
- Two highways
- 1 1/2 hours of camera footage
- Filter out trucks
- Extract left two lanes

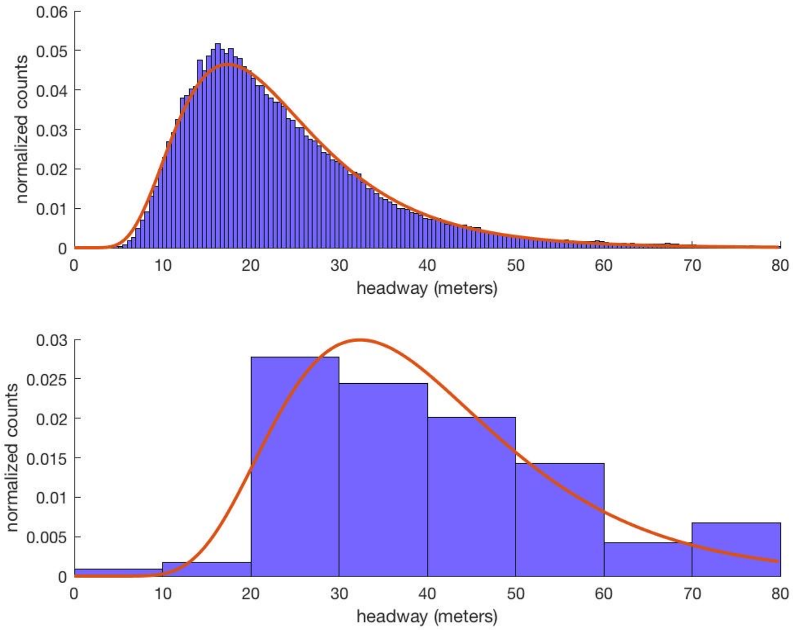




Model Calibration - Probabilities

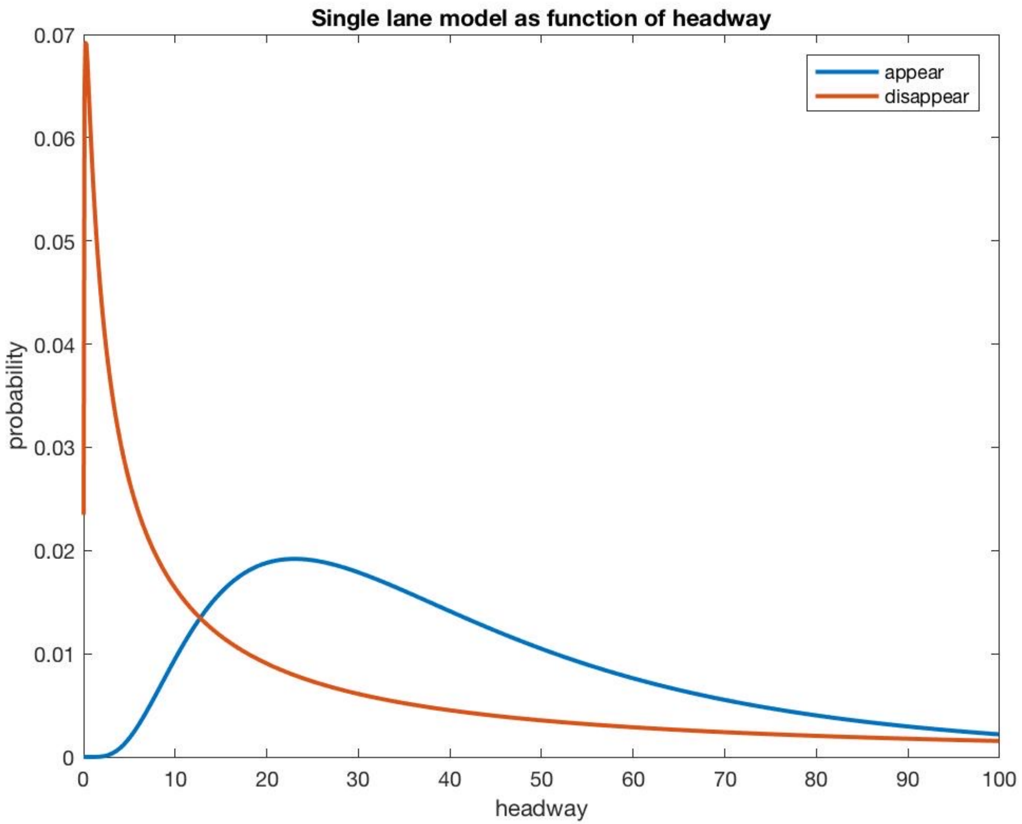
- Extract headways when lane changes occur
- Fit to log-normal distribution
- $p(lc|h) = \frac{p(h|lc)*p(lc)}{p(h)}$





Fit of log-normal distribution for the total distribution of headways and the conditional distribution of headways when a vehicle lane changes into a lane, respectively, computed for 7:50 am on the US 101.

Single lane model



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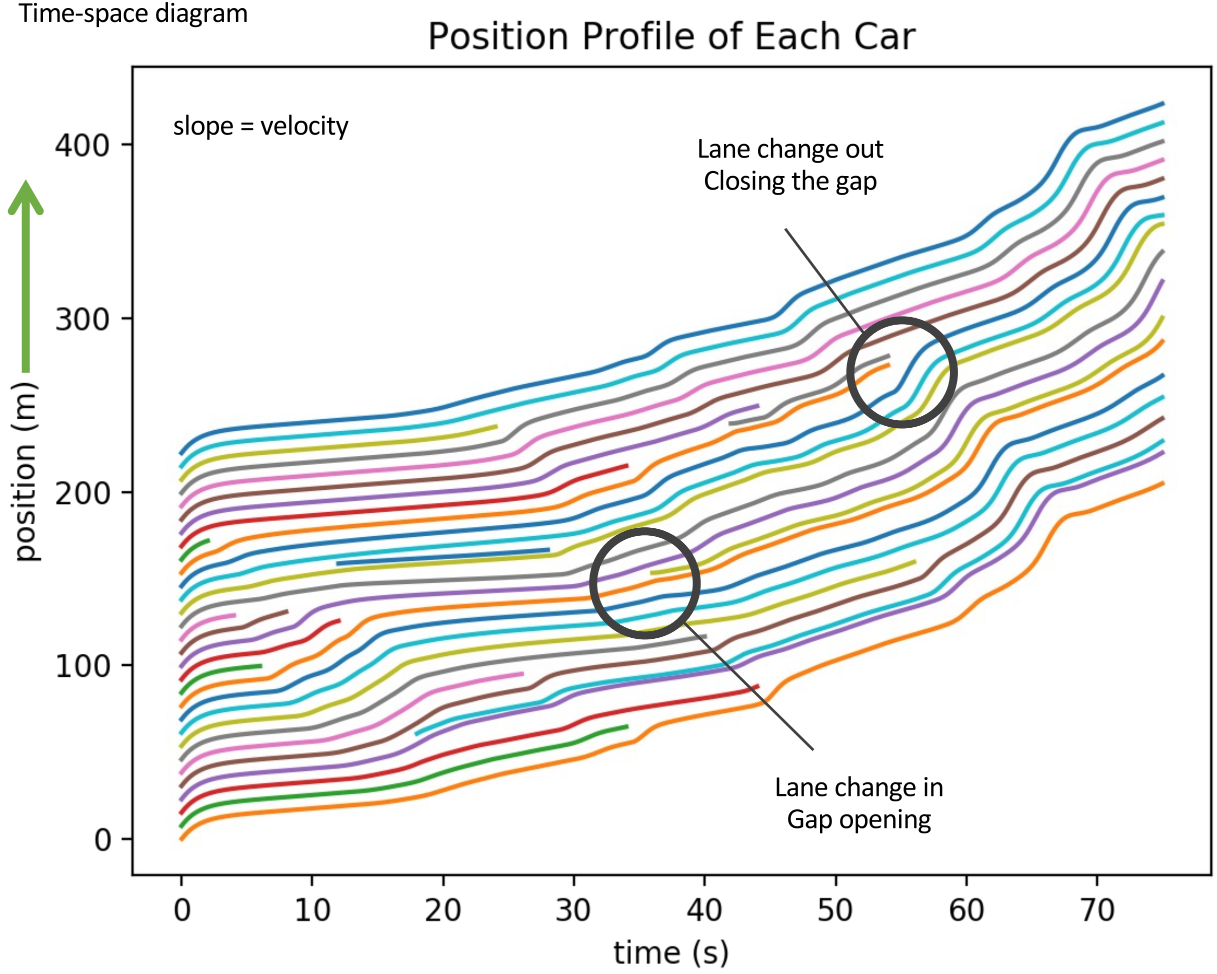
Time-space diagram



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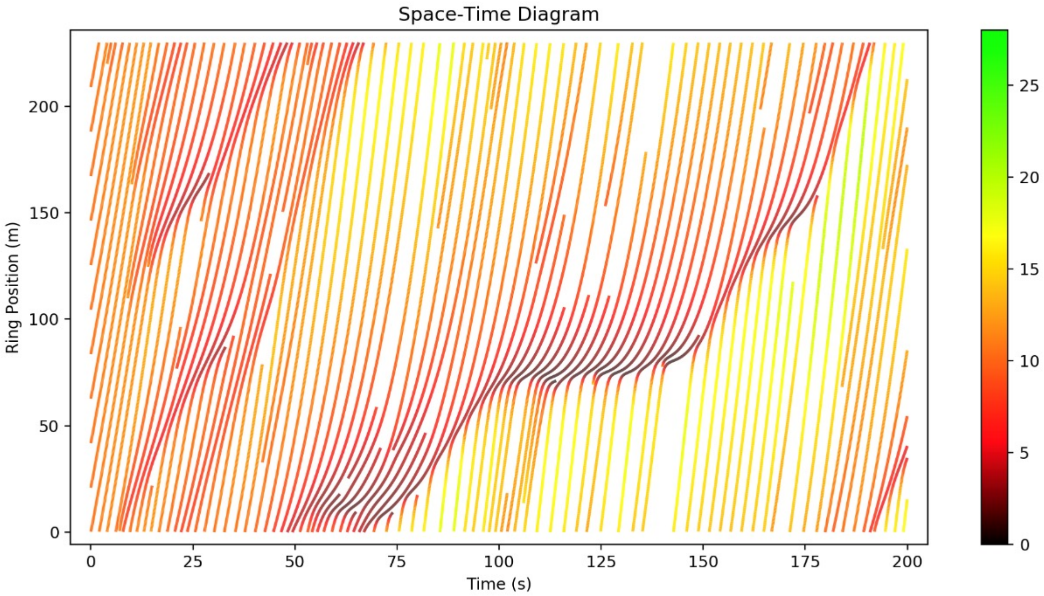
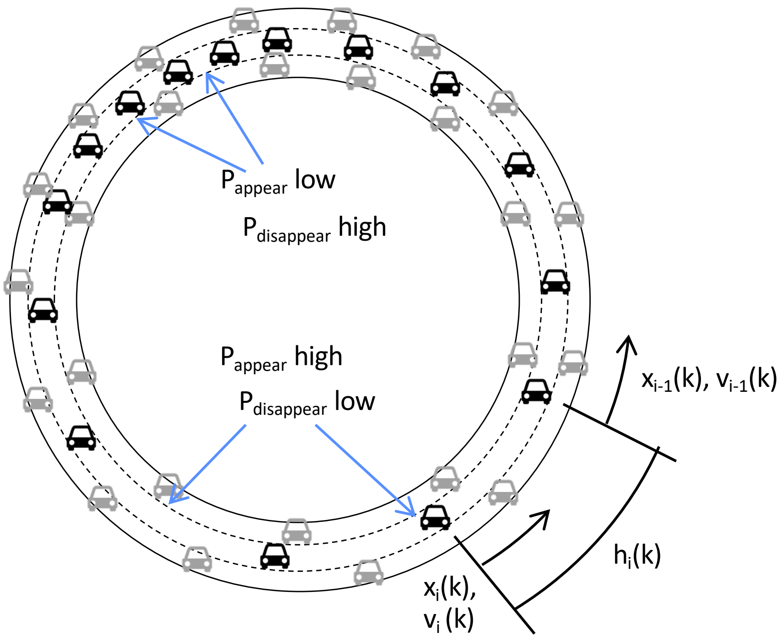


Time step (deterministic car following) = 0.025 sec
Time step (stochastic lane changing) = 6 sec

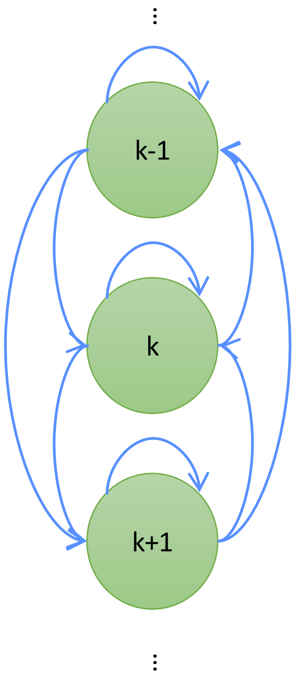
Wu, Cathy, Eugene Vinitsky, Aboudy Kreidieh, and Alexandre Bayen. "Multi-lane reduction: A stochastic single-lane model for lane changing." IEEE ITSC, 2017.

Two reductions

Stochastic single-lane model



Macroscopic multi-lane traffic



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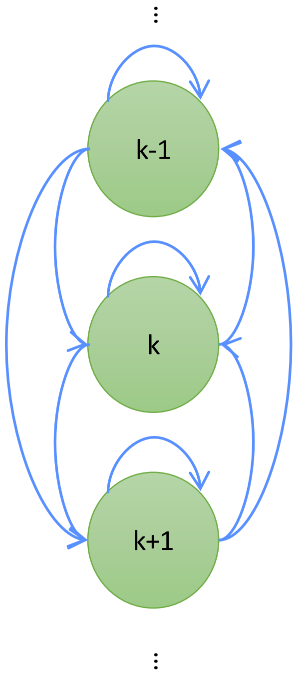
Macroscopic multi-lane model

$i + \Delta$ vehicles appear

$$T_{n,n'} = \sum_{i=0}^{n-\Delta} \binom{n}{i+\Delta} [p_a(h_n^*)]^{\Delta+i} [1 - p_a(h_n^*)]^{n-(\Delta+i)} \times \binom{n}{i} [p_d(h_n^*)]^{i} [1 - p_d(h_n^*)]^{n-i}$$

i vehicles disappear

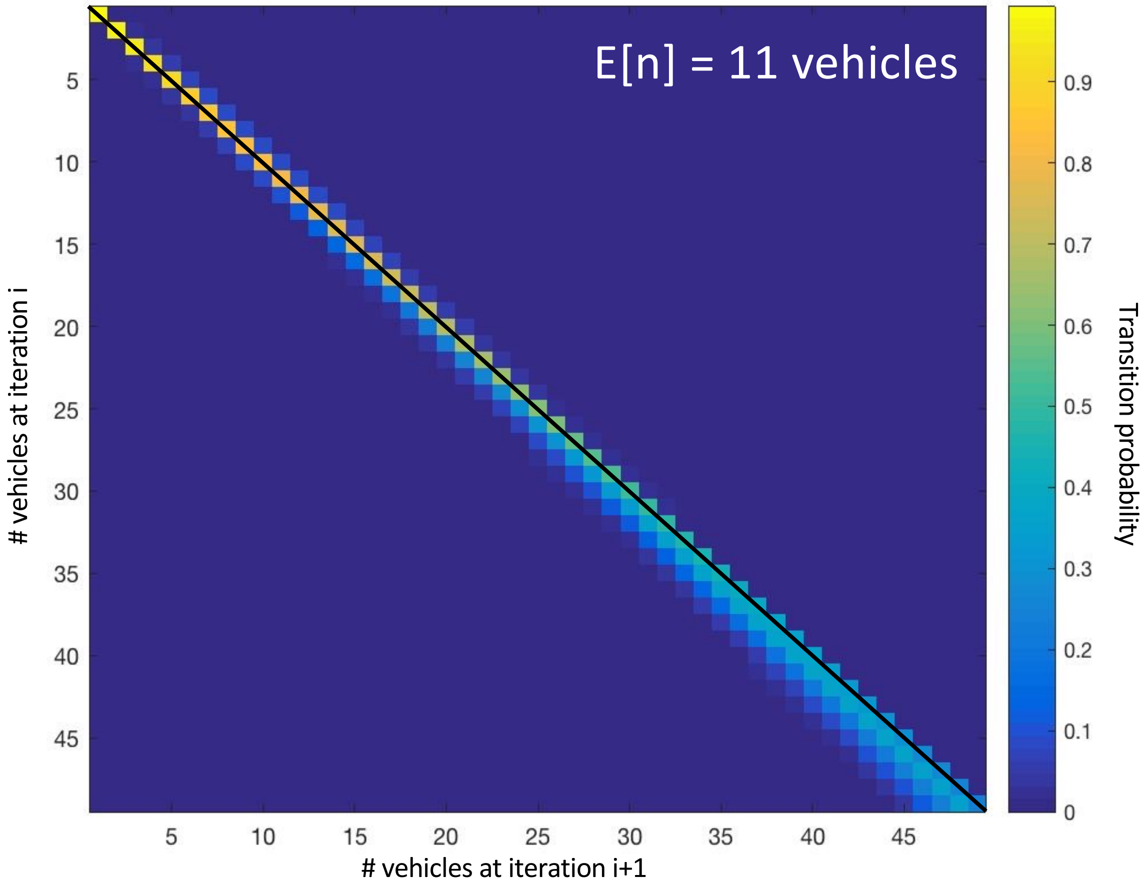
$$\Delta := n' - n$$



Theorem (Stationary)

This Markov chain converges to a stationary distribution.

Transition matrix for the number of vehicles in the lane



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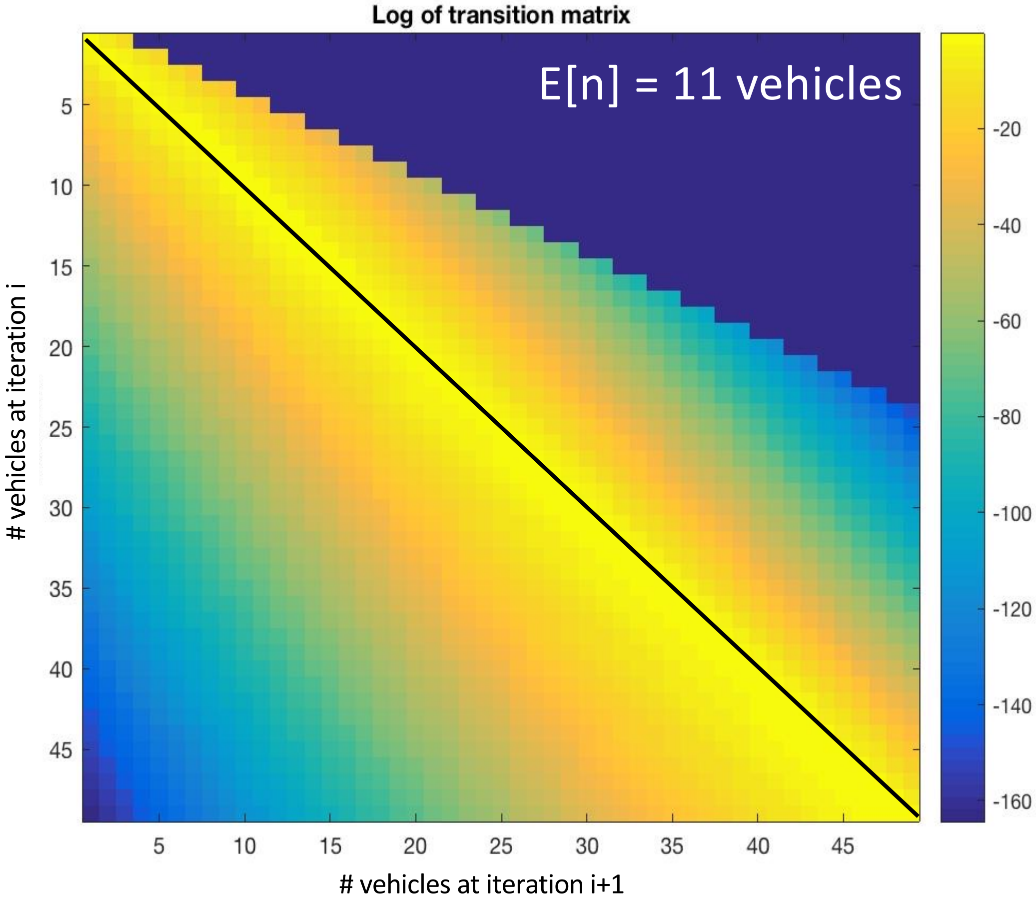
Transition matrix for the number of vehicles in the lane



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Variance analysis

Observation: Variance of velocity is related to energy consumption.

Experiment Type	σ_h^2	σ_v^2	
Lane Changing Simulation	61	80	
No Lane Changing Simulation	58	97	(Does not model lane changing)
Markov Chain	61	41	(Does not model transients, stop-and-go traffic)

Variance of headway

Variance of velocity

Insight: Discretionary lane changes may aid in reducing stop-and-go traffic waves rather than increase them.

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References

1. Prof Ayalvadi Ganesh. “Markov Chains.” Lecture notes. https://people.maths.bris.ac.uk/~maajg/teaching/pgt/markov_chains.pdf

2. Wu, Cathy, Eugene Vinitsky, Aboudy Kreidieh, and Alexandre Bayen. "Multi-lane reduction: A stochastic single-lane model for lane changing." In *International Conference on Intelligent Transportation Systems (ITSC)*, 2017.

3. Slides adapted from Carolina Osorio