Markov chains

Computing facility dynamics

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1.041/1.200 Transportation: Foundations and Methods

Readings

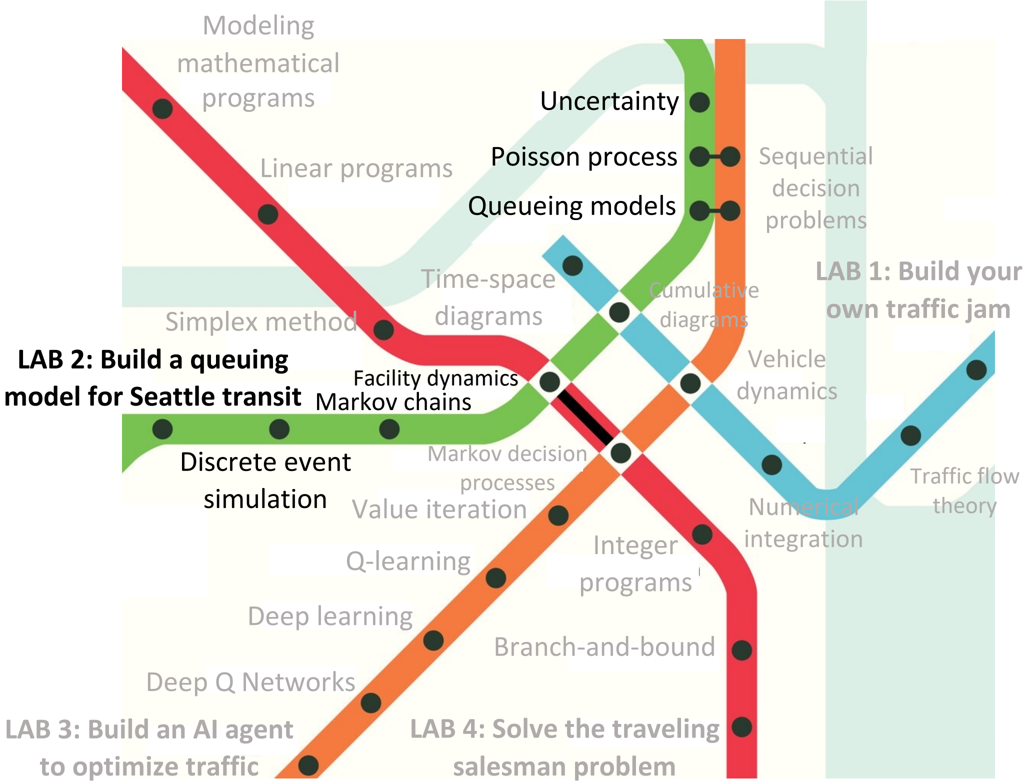
1. (Optional) Prof Ayalvadi Ganesh. “Markov Chains.” Lecture notes. URL.

Unit 2: Queuing systems

Unit 2

Modeling

Stochastic



Outline

1. Transition rate matrix
2. Markov chains

Outline

1. **Transition rate matrix**
   a. Global balance equations
   b. Container terminal problem
2. Markov chains

Beyond the basic queues

- In this lecture, we discuss a more general strategy for modeling and analyzing more sophisticated facility dynamics (queues).
  - We will introduce ideas of: transition rate matrix, global balance equations. We will leverage fundamental results from Markov chains.
  - Goal is still to derive $\overline{N}, \overline{N}_q, \overline{T}, \overline{T}_q$
- Solutions are no longer analytical, but rather computed numerically.
  - For now, we will still use properties of the Exponential distribution.
  - In Lecture 10, we will model even more general queues (see Computational Lab #2).
- Why do we need fancier tools?

Transition probability matrix

- Consider a sequence $\{X_t, t \geq 0\}$ that goes from state i to j with probability τ_{ij}
  - $T = \left(\tau_{ij}\right)_{i\in I, j\in J}$ is the transition **probability** matrix
- Define $P(k)$ as probability distribution of the state of the system after k steps, i.e., $P(k)_i = P(X_{k\Delta t} = i)$
- Evolution: $P(k+1)^T \approx P(k)^T T$, where $T := I + Q\Delta\mathrm{t}$
  - Q is the transition **rate** matrix; for Δt sufficiently small

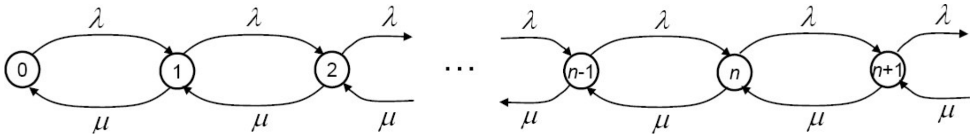
Transition probability matrix

Transition rate matrix

- Define transition rate matrix $Q = \left(q_{ij}\right)_{i \in I, j \in J}, = \begin{bmatrix} q_{00} & q_{01} & \cdots \\ q_{10} & q_{11} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$
  - where q_{ij} is transition rate from state i to state j
  - $q_{ii} = -\sum_{j \neq i} q_{ij}$: rate of departure from state i
  - Square matrix
  - Helps describes the rate by which the system changes state, i.e., $\frac{d}{dt} P_n(t)$
- Possible transitions from a given state i, with their corresponding rates and conditions under which these transitions can take place

Example: M/M/1

initial state s	new state j	rate q_{ij}	condition
n	$n+1$	λ	always possible
n	$n-1$	μ	$n \geq 1$



Global balance equations (stationary analysis!)

- The set of equations represented by the state transition diagram are called the global balance equations. They can be written as:

$$\sum_{j \neq i} P_j q_{ji} = -P_i q_{ii} \quad \forall i \quad \underset{\text{(equivalent)}}{\equiv} \quad P^T Q = 0$$

- In other words, in steady state (when $P(t) = P$), transitions into and out of states must be balanced.
- In some cases (e.g., M/M/c), the global balance equations can be directly solved for P (stationary analysis).
- In other cases…

Stationary analysis: in a nutshell

- Stationary distribution: often denoted P or π
- Four step approach:
  1. Define the state space
  2. Define the state transition diagram or the **transition rate matrix**
  3. Then, derive the balance equations to obtain the stationary distribution
  4. Derive $\bar{N}$, $\bar{N}_q$, $\bar{T}$, $\bar{T}_q$

Container terminal

- Arrival
  - Ships arrive at a container terminal according to a Poisson process with rate λ. Each ship brings thousands of containers.
  - There are c berths, i.e. only c ships can dock or anchor at the port simultaneously.
  - If all berths are occupied arriving ships must wait at the entrance of the port. A maximum of r ships can wait for entrance, further ships are redirected to another port.
- Berth operations
  - Upon arrival to the berth, a ship unloads its containers. The unloading time follows an exponential distribution with parameter μ.

- Departure
  - Once all containers are unloaded, the ship can leave the port, as long as the exit is clear, which occurs with probability p_f.
  - Suppose each berth has its own exit. If the exit is occupied, it remains so during an exponentially distributed time with parameter β.

Container terminal

Upon arrival to the port a ship (arrival rate λ):

1. [queue]
2. is served (service rate μ)
3. [blocked] (rate at which exit is occupied β, also exponentially distributed)
4. Departs (with probability p_f)

Note: Blocking after service: blocking mechanism used to describe how congestion arises and propagates

State space of port :

$$S = \{(A, B, W) \in N^3, A + B \leq c, W \leq r\}$$

- A = ships being served
- B = ships waiting to depart
- W = ships waiting to be served

Stationary distribution:

$$P = \big(P\big((A, B, W) = (a, b, w)\big), (a, b, w) \in S\big)$$

Problem: For $c = 2, r = 2$, define the possible states, transitions, & corresponding rates (Q matrix).

Reminder (Departure): Once all containers are unloaded, the ship can leave the port, as long as the exit is clear, which occurs with probability p_f. Suppose each berth has its own exit. If the exit is occupied, it remains so during an exponentially distributed time with parameter β.

Container terminal

Outline

1. Transition rate matrix
2. **Markov chains**
   a. Python demo

Markov chain

- The sequence $\{X_t, t \geq 0\}$ that goes from state i to j with probability τ_{ij}, independently of the states visited before, is a Markov chain.
- τ_{ij} is also called a transition probability.
- Markov property: the current state contains all information for predicting the future of the process/chain.

Preview (Unit 4): Markov decision processes (MDP)

- Extension of Markov chains, where, in addition to the current state, the transition is also affected by **decisions (control)**, i.e. τ_{ijk}.
- Example of sequential decisions: traffic signal phase change, autonomous vehicle steering & acceleration, bus holding decisions, dynamic speed limits, rideshare matching

Ergodic Theorem for Markov chains

Distribution after k steps (each Δt, i.e. $t = \Delta t\ k$)

$$P_n(k+1) = \sum_{i \in N} P_i(k)\tau_{in}$$

Stationary distribution

$$P_n = \sum_{i \in N} P_i \tau_{in}$$

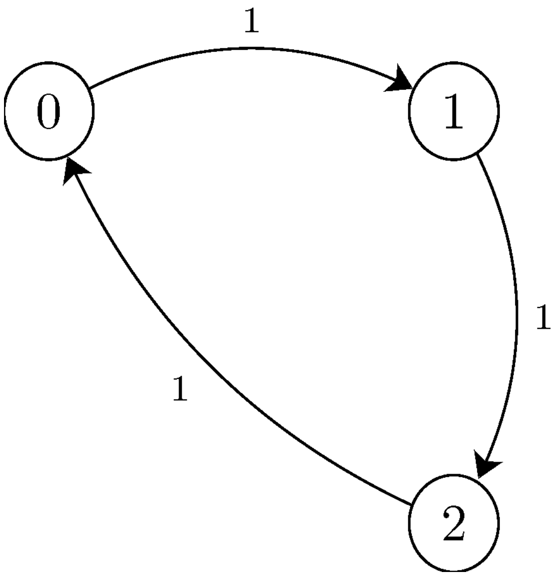
Definitions:

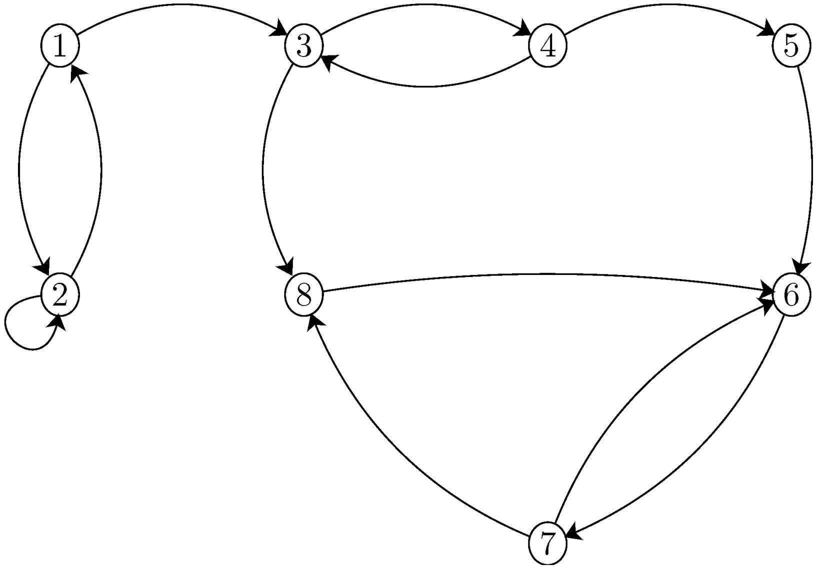
- **Irreducible**: can reach any state from any state, possibly after many steps
- **Aperiodic**: no state transitions back to itself **only** under cycle lengths of integer multiples >1
- **Positive Recurrent**: the expected time to return to any state is finite
  - For an M/M/1 queue, this is satisfied only if $\rho < 1$.

Theorem (Ergodic Theorem of Markov Chains)

1. If the Markov chain is irreducible and positive recurrent, it has a unique stationary distribution P and P_n is the long-term fraction that $X(t) = n$.
2. If the chain is also aperiodic, then the distribution $P(k)$ of $X(t)$ converges to P as $k \to \infty$.

Example: Markov chains





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   a. **Python demo**

References

1. Prof Ayalvadi Ganesh. "Markov Chains." Lecture notes. https://people.maths.bris.ac.uk/~maajg/teaching/pgt/markov_chains.pdf

2. Slides adapted from Carolina Osorio