

SURFACE TENSION MODULE

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This set of notes has been developed as supporting material for the Surface Tension module in the 1.63J/2.21J Fluid Dynamics course, which will be presented in a series of 6 lectures at the end of the spring term. These lecture notes have been drawn from many sources, including text books, journal articles, and lecture notes from courses taken by the author as a student. These notes are not intended as a complete discussion of the subject, or as a scholarly work in which all relevant references are cited. Rather, they are intended as an introduction that will hopefully motivate the interested student to learn more about the subject. Topics have been chosen according to their perceived value in developing the physical insight of the students.

LECTURE 1: The definition and scaling of surface tension

1.1 Surface tension: a working definition

Discussions of the molecular origins of surface or interfacial tension may be found elsewhere (e.g. Israelachvili 1995, Rowlinson & Widom 1982). Our discussion follows that of De Gennes, Brochard-Wyart & Quéré (2003).

Molecules in a fluid feel a mutual attraction. When this attractive force is overcome by thermal agitation, the molecules pass into a gaseous phase. Let us first consider a free surface, for example that between air and water. A water molecule in the fluid bulk is surrounded by attractive neighbours, while a molecule at the surface is attracted by a reduced number of neighbours and so in an energetically unfavourable state. The creation of new surface is thus energetically costly, and a fluid system will act to minimize surface areas. It is thus that small fluid bodies tend to evolve into spheres; for example, a thin fluid jet emerging from your kitchen sink will generally pinch off into spherical drops in order to minimize the total surface area (see Lecture 5).

If U is the total cohesive energy per molecule, then a molecule at a flat surface will lose $U/2$. Surface tension is a direct measure of this energy loss per unit area of surface. If the characteristic molecular dimension is R and its area thus R^2 , then the surface tension is $\sigma \sim U/(2R^2)$. Note that surface tension increases as the intermolecular attraction increases and the molecular size decreases. For most oils, $\sigma \sim 20$ dynes/cm, while for water, $\sigma \sim 70$ dynes/cm. The highest surface tensions are for liquid metals; for example, liquid mercury has $\sigma \sim 500$ dynes/cm.

The origins of interfacial tension are analogous. Interfacial tension is a material property of a fluid-fluid interface whose origins lie in the different attractive intermolecular forces that act in the two fluid phases. The result is an interfacial energy per area that acts to resist the creation of new interface, and that is equivalent to a line tension acting in all directions parallel to the interface. Fluids between which no interfacial tension arises are said to be miscible. For example, salt molecules will diffuse freely across a boundary between fresh and saltwater; consequently, these fluids are miscible, and there is no interfacial tension between them. Our discussion will be confined to immiscible fluid-fluid interfaces (or fluid surfaces), at which an effective interfacial (or surface) tension acts.

Surface tension σ has the units of force/length or equivalently energy/area, and so may be thought of as a negative surface pressure. Pressure is generally an isotropic force per area that acts through-

out the bulk of a fluid: a small surface element dS will feel a total force $p(\mathbf{x})dS$ owing to the local pressure field $p(\mathbf{x})$. If the surface S is closed, and the pressure uniform, the net pressure force acting on S is zero and the fluid remains static. Pressure gradients correspond to body forces (with units of force per unit volume) within a fluid, and so appear explicitly in the Navier-Stokes equations. Surface tension has the units of force per length, and its action is confined to the free surface. Consider for the sake of simplicity a perfectly flat interface. A surface line element $d\ell$ will feel a total force $\sigma d\ell$ owing to the local surface tension $\sigma(\mathbf{x})$. If the surface line element is a closed loop C , and the surface tension uniform, the net surface tension force acting on C is zero, and the fluid remains static. If surface tension gradients arise, there may be a net force on the surface element that acts to distort it through driving flow.

1.2 Governing Equations

The motion of a fluid of uniform density ρ and viscosity μ is governed by the Navier-Stokes equations, which represent a continuum statement of Newton's laws.

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mathbf{F} + \mu \nabla^2 \mathbf{u} \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (2)$$

This represents a system of 4 equations in 4 unknowns (the fluid pressure p and the three components of the velocity field \mathbf{u}). Here \mathbf{F} represents any body force acting on the fluid; for example, in the presence of a gravitational field, $\mathbf{F} = \rho \mathbf{g}$, where \mathbf{g} is the acceleration due to gravity.

Surface tension acts only at the free surface; consequently, it does not appear in the Navier-Stokes equations, but rather enters through the boundary conditions. The boundary conditions appropriate at a fluid-fluid interface are formally developed in Lecture 2. We here simply state them for the simple case of a free surface (such as air-water, in which one of the fluids is not dynamically significant) in order to get a feeling for the scaling of surface tension.

The normal stress balance at a free surface must be balanced by the curvature force associated with the surface tension:

$$\mathbf{n} \cdot \mathbf{T} \cdot \mathbf{n} = \sigma (\nabla \cdot \mathbf{n}) \quad (3)$$

where $T = -p\mathbf{I} + \mu[\nabla\mathbf{u} + (\nabla\mathbf{u})^T] = -p\mathbf{I} + 2\mu\mathbf{E}$ is the stress tensor, $\mathbf{E} = 1/2 [\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$ is the deviatoric stress tensor, and \mathbf{n} is the unit normal to the surface. The tangential stress at a free surface must balance the local surface tension gradient:

$$\mathbf{n} \cdot \mathbf{T} \cdot \mathbf{t} = \nabla \sigma \cdot \mathbf{t} \quad (4)$$

where \mathbf{t} is the unit tangent to the interface.

1.3 The scaling of surface tension

We consider a fluid of density ρ and viscosity $\mu = \rho\nu$ with a free surface characterized by a surface tension σ . The flow is marked by characteristic length- and velocity-scales of, respectively, a and U and evolves in the presence of a gravitational field $\mathbf{g} = -g\hat{z}$. We thus have a physical system defined in terms of six physical variables ($\rho, \nu, \sigma, a, U, g$) that may be expressed in terms of three

fundamental units: mass, length and time. Buckingham's Theorem thus indicates that the system may be uniquely prescribed in terms of three dimensionless groups. We choose

$$Re = \frac{Ua}{\nu} = \frac{INERTIA}{VISCOSITY} = \text{Reynolds number} \quad (5)$$

$$Fr = \frac{U^2}{ga} = \frac{INERTIA}{GRAVITY} = \text{Froude number} \quad (6)$$

$$Bo = \frac{\rho ga^2}{\sigma} = \frac{GRAVITY}{CURVATURE} = \text{Bond number} \quad (7)$$

The Reynolds number prescribes the relative magnitudes of inertial and viscous forces in the system, while the Froude number those of inertial and gravity forces. The Bond number indicates the relative importance of forces induced by gravity and surface tension. Note that these two forces are comparable when $Bo = 1$, which arises on a lengthscale corresponding to the capillary length: $\ell_c = (\sigma/(\rho g))^{1/2}$. For an air-water surface, for example, $\sigma \approx 70$ dynes/cm, $\rho = 1$ g/cc and $g = 980$ cm/s², so that $\ell_c \approx 2$ mm. Bodies of water in air are dominated by the influence of surface tension provided they are smaller than the capillary length. Roughly speaking, the capillary length prescribes the maximum size of pendant drops that may hang inverted from a ceiling, water-walking insects, and raindrops. Note that as a fluid system becomes progressively smaller, the relative importance of surface tension and gravity increases; it is thus that surface tension effects are dominant in microscale engineering processes.

Finally, we note that other frequently arising dimensionless group may be formed from the product of B , Re and Fr :

$$We = \frac{\rho U^2 a}{\sigma} = \frac{INERTIA}{CURVATURE} = \text{Weber number} \quad (8)$$

$$Ca = \frac{\rho \nu U}{\sigma} = \frac{VISCOSUS}{CURVATURE} = \text{Capillary number} \quad (9)$$

The Weber number indicates the relative magnitudes of inertial and curvature forces within a fluid, and the capillary number those of viscous and curvature forces. Finally, we note that if the flow is marked by a Marangoni stress of characteristic magnitude $\Delta\sigma/L$, then an additional dimensionless group arises that characterizes the relative magnitude of Marangoni and curvature stresses: $a\Delta\sigma/(L\sigma)$.

We now demonstrate how these dimensionless groups arising naturally from the nondimensionalization of Navier-Stokes equations and the surface boundary conditions. We first introduce a dynamic pressure: $p_d = p - \rho \mathbf{g} \cdot \mathbf{x}$, so that gravity appears only in the boundary conditions. We consider the special case of high Reynolds number flow, for which the characteristic dynamic pressure is ρU^2 . We define dimensionless primed variables according to:

$$\mathbf{u} = U \mathbf{u}' \quad , \quad p_d = \rho U^2 p'_d \quad , \quad \mathbf{x} = a \mathbf{x}' \quad , \quad t = \frac{a}{U} t' \quad (10)$$

Nondimensionalizing the Navier-Stokes equations and appropriate boundary conditions yields the following system:

$$\left(\frac{\partial \mathbf{u}'}{\partial t'} + \mathbf{u}' \cdot \nabla' \mathbf{u}' \right) = -\nabla' p'_d + \frac{1}{Re} \nabla'^2 \mathbf{u}' \quad , \quad \nabla' \cdot \mathbf{u}' = 0. \quad (11)$$

The normal stress condition assumes the dimensionless form:

$$-p'_d + \frac{1}{Fr} z' + \frac{2}{Re} \mathbf{n} \cdot \mathbf{E}' \cdot \mathbf{n} = \frac{1}{We} \nabla' \cdot \mathbf{n} \quad (12)$$

The importance of surface tension relative to gravity and viscous stresses is prescribed by the relative magnitudes of the Weber, Froude and Reynolds numbers. In the high Re limit of interest, the normal force balance requires that the dynamic pressure be balanced by either gravitational or curvature stresses, the relative magnitudes of which are prescribed by the Bond number.

The nondimensionalization scheme will depend on the physical system of interest. Our purpose here was simply to illustrate the manner in which the dimensionless groups arise in the theoretical formulation of the problem. Moreover, we see that those involving surface tension enter exclusively through the boundary conditions.

References

While this list of relevant textbooks is far from complete, we include it as a source of additional reading for the interested student.

Rowlinson, J.S. and Widom, B., 1982, *Molecular theory of capillarity*, Dover.

Israelachvili, J., 1995. *Intermolecular and surface forces*, 2nd Edition, Academic Press.

DeGennes, P., Brochard-Wyart, M. and Quere, D., 2002. *Perles, Gouttes, bulles, perles et ondes*, Belin.