Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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February 6, 2007
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1.8 Rayleigh's Problem - solid wall as a source of vorticity

Owing to terms representing convective inertia, the Navier-Stokes equations are highly nonlinear. Explicit solutions are usually limited to a class of problems where inertia is identically zero. This happens when the flow is unidirectional and uniform. Flow quantities depend only on a transverse coordinate. We discuss one such example with a view to examining the role of viscosity.

Consider a two-dimensional flow in the upper half plane of (x, y) bounded below by a rigid plate coinciding with the x axis. At $t = 0$ the plate suddenly moves in the tangential direction at constant velocity U. Find the development of the fluid motion in the region $y > 0$.

Because the plate is infinite in extent, the flow must be uniform in x, i.e. $\frac{\partial}{\partial x} = 0$. It follows from continuity that

$$\frac{\partial v}{\partial y} = 0, \quad y > 0$$

implying that $v =$ constant in y. Since $v(0, t) = 0$, $v \equiv 0$ for all y. Therefore, the only unknown is $u(y, t)$ which must satisfy the momentum equation,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{1.8.1}$$

where

$$\nu = \frac{\mu}{\rho} \tag{1.8.2}$$

denotes the kinematic viscosity. The boundary conditions are :

$$u = U,\ y = 0,\ t > 0; \quad \text{no slip} \tag{1.8.3}$$

$$u = 0,\ y \sim \infty,\ t > 0 \tag{1.8.4}$$

The initial condition is

$$u = 0,\ t = 0,\ \forall y \tag{1.8.5}$$

Mathematically this is the heat conduction problem for a semi-infinite rod. The solution is well-known (Carlaw & Jeager, *Conduction of Heat in Solids* or Mei, *Mathematical Analysis in Engineering*) ,

$$u = U\left(1 - \text{erf}\frac{y}{2\sqrt{\nu t}}\right) \tag{1.8.6}$$

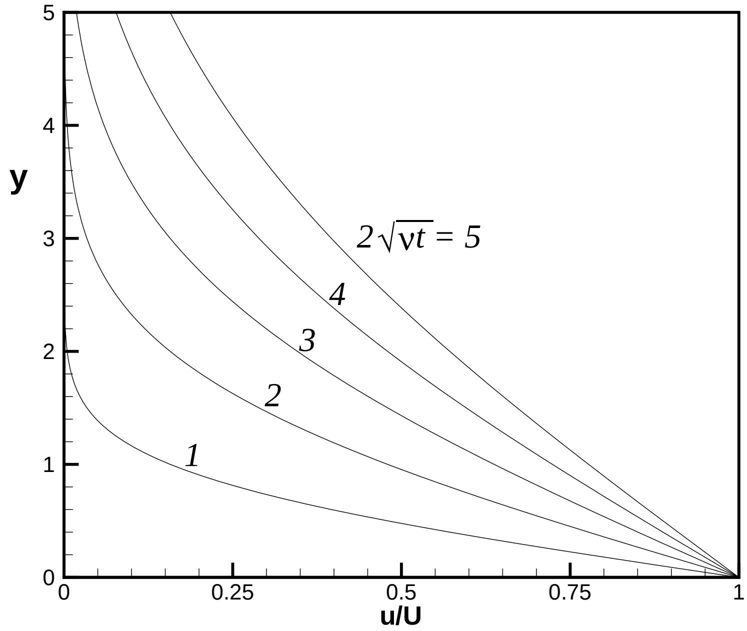

Figure 1.8.1: Velocity profile due to impulsive motion of x-plane

where

$$\operatorname{erf} \zeta = \frac{2}{\sqrt{\pi}} \int_0^{\zeta} e^{-\lambda^2} \, d\lambda. \tag{1.8.7}$$

is the error function. As shown in Figure 1.8.1,

fluid momentum is diffused away from the plane $y = 0$. The region affected by viscosity (the boundary layer) grows in time as $\delta \sim \sqrt{\nu t}$. This observation can be confirmed, indeed anticipated, merely by a scaling argument based on the momentum equation (1.8.1) without solving it. Let U, t, δ denote the scales of velocity, time and region of viscosity respectively. For viscosity to be important, the two terms in (1.8.1) must be comparable in order of magnitude, i.e.,

$$\frac{U}{t} \sim \nu \frac{U}{\delta^2}$$

It follows that

$$\delta \sim \sqrt{\nu t}$$

Let us use this simple example to study the role of vorticity

$$\vec{\zeta} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \vec{k} \tag{1.8.8}$$

In this problem there is only one vorticity component,

$$\zeta_3 = -\frac{\partial u}{\partial y} = U\,\frac{\partial}{\partial y}\,\mathrm{erf}\,\frac{y}{2\sqrt{\nu t}} = \frac{2U}{\sqrt{4\pi\nu t}}\,e^{-y^2/4\nu t}. \tag{1.8.9}$$

which is just the velocity shear. Mathematically (1.8.9) is the solution to the diffusion equation

$$\frac{\partial \zeta}{\partial t} = \nu\,\frac{\partial^2 \zeta}{\partial y^2}. \tag{1.8.10}$$

which follows from (1.8.1), and the initial condition that there is a plane source of at $y = 0$:

$$\zeta_3(y, 0) = 2U\delta(y). \tag{1.8.11}$$

Thus vorticity is diffused away from the solid wall which acts as a voriticity source. Note that the shear stress at the wall is

$$\tau_{xy}(0, t) = \mu\,\frac{\partial u}{\partial y}\bigg|_{y=0} = -\rho U\,\sqrt{\frac{\nu}{\pi t}}. \tag{1.8.12}$$

which is initially infinite but decays with time.

Why is the wall a source of vorticity? Just after the plane started to move there is a velocity discontinuity at $y = 0+$. The associated velocity gradient is $\partial u/\partial y = -U\delta(y)$ hence the vorticity is a highly concentrated function of y: $-\partial u/\partial y = U\delta(y)$. Furthermore the half space $(0 < y < \infty)$ problem can be thought of as one half of the whole plane problem for $-\infty < y < \infty$ if the top of the fluid in the lower half plane suddenly moves to the left at the speed U. This would give an initial vorticity $U\delta(y)$ at $y = 0-$. Thus for the a whole space problem there is a vorticity source of total strength $2U\delta(y)$ at the initial instant. As time proceeds, half of the released vorticity is diffused to the region of $y > 0$ and half to $y < 0$. Thus, the solid wall is the source of vorticity.

The reader can verify the solution (1.8.9) by assuming a similarity form,

$$\zeta_3(y, t) = \frac{C}{\sqrt{t}} f\left(\frac{y}{\sqrt{t}}\right) \tag{1.8.13}$$