

Lecture Notes on Fluid Dynamics

(1.63J/2.21J)

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2-3-gr-current.tex

References:

C. C. Mei, (1966), *J. Math. & Phys.* pp.

K. F. Liu & C. C. Mei, 1989, *J Fluid Mech*, (207) 505-529. Slow flow of a Bingham-plastic fluid down an incline.

2.3 A gravity current

For the highly nonlinear equation, a relatively simple solution is that of a stationary (or permanent) wave which is profile advancing at a constant speed without changing its shape. Mathematically the profile is describable as

$$h(x, t) = h(x - Ct) = h(\sigma), \quad \sigma = x - Ct \quad (2.3.1)$$

By the chain rule of differentiation,

$$\frac{\partial h(x - Ct)}{\partial t} = \frac{dh}{d\sigma} \frac{\partial \sigma}{\partial t} = -C \frac{dh}{d\sigma}, \quad \frac{\partial h(x - Ct)}{\partial x} = \frac{dh}{d\sigma} \frac{\partial \sigma}{\partial x} = \frac{dh}{d\sigma}$$

Hence (2.3.30) reduces to an ordinary differential equation,

$$-C \frac{dh}{d\sigma} + \frac{\rho g \cos \theta}{3\mu} \frac{d}{d\sigma} \left[h^3 \left(\tan \theta - \frac{dh}{d\sigma} \right) \right] = 0 \quad (2.3.2)$$

Integrating once we get

$$-Ch + \frac{\rho g \cos \theta}{3\mu} \left[h^3 \left(\tan \theta - \frac{dh}{d\sigma} \right) \right] = \text{constant}$$

Let the gravity current advance along a dry bed, then $h = 0$ is a part of the solution. The constant of integration must be set to zero. Introducing the dimensionless variables

$$h = H_c h', \quad \sigma = L_c \sigma', \quad \text{with } L_c = H_c / \tan \theta, \quad (2.3.3)$$

where H_c is the maximum depth far upstream, we get

$$-\frac{3C\mu}{\rho g H_c^2 \sin \theta} h' + h'^3 \left(1 - \frac{dh'}{d\sigma'} \right) = 0, \quad (2.3.4)$$

Let the gravity current be uniform far upstream, then

$$h' \rightarrow 1, \quad \frac{dh'}{d\sigma'} \rightarrow 0, \quad \text{as } \sigma' \rightarrow -\infty. \quad (2.3.5)$$

It follows that

$$\frac{3C\mu}{\rho g H_c^2 \sin \theta} = 1$$

or,

$$\boxed{C = \frac{\rho g H_c^2 \sin \theta}{3\mu}} \quad (2.3.6)$$

and

$$h' \left[-1 + h'^2 \left(1 - \frac{dh'}{d\sigma'} \right) \right] = 0, \quad (2.3.7)$$

One of the solution is $h' = 0$, representing the dry bed. For the nontrivial solution, we rewrite

$$d\sigma' = -\frac{h^2 dh}{1-h^2} = dh \left[1 - \frac{1}{2} \left(\frac{1}{1-h} + \frac{1}{1+h} \right) \right] \quad (2.3.8)$$

which can be integrated to give

$$\boxed{h' + \frac{1}{2} \log \left(\frac{1-h'}{1+h'} \right) = \sigma' - \sigma'_o} \quad (2.3.9)$$

This is an implicit relation between h' and σ' , and represents a smooth surface decreasing monotonically from $h = 1$ at $\sigma' \sim -\infty$ to $h' = 0$ at the front $\sigma' = \sigma'_o$, as plotted in Figure 2.3.1. Note from (2.3.8) that $d\sigma'/dh' = 0$ when $h' = 0$, implying infinite slope at the tip of the gravity current. This infinity violates the original approximation that $dh'/d\sigma' = O(1)$. Fortunately it is highly localized and does not affect the validity of the theory elsewhere (see Liu & Mei, 1989, JFM).

Eq. (2.3.6) tells us that the speed of the front is higher for a thicker layer, steeper slope or smaller viscosity. This relation can be confirmed by a quicker argument. In the fixed frame of reference, the total flux must be equal to CH . therefore C must be equal to the depth-averaged velocity \bar{u} which is given by (2.3.19) with $\partial h/\partial x = 0$.

A similar analysis has been applied to a fluid-mud which is non-Newtonian characterized by the yield stress. Laboratory simulations have been reported by Liu & Mei (*J. Fluid Mech.* 207, 505-529.) who used a kaolinite/water mixture. Figure 2.3.2 shows the setup of the inclined flume and Figure 2.3 shows the recorded profiles of the gravity current along with the theory. The agreement is very good, despite the steep front where the approximation is locally invalid.

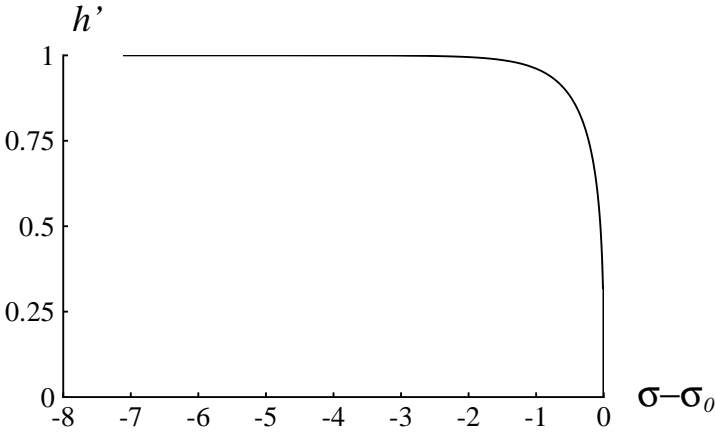


Figure 2.3.1: Gravity current down an inclined plane

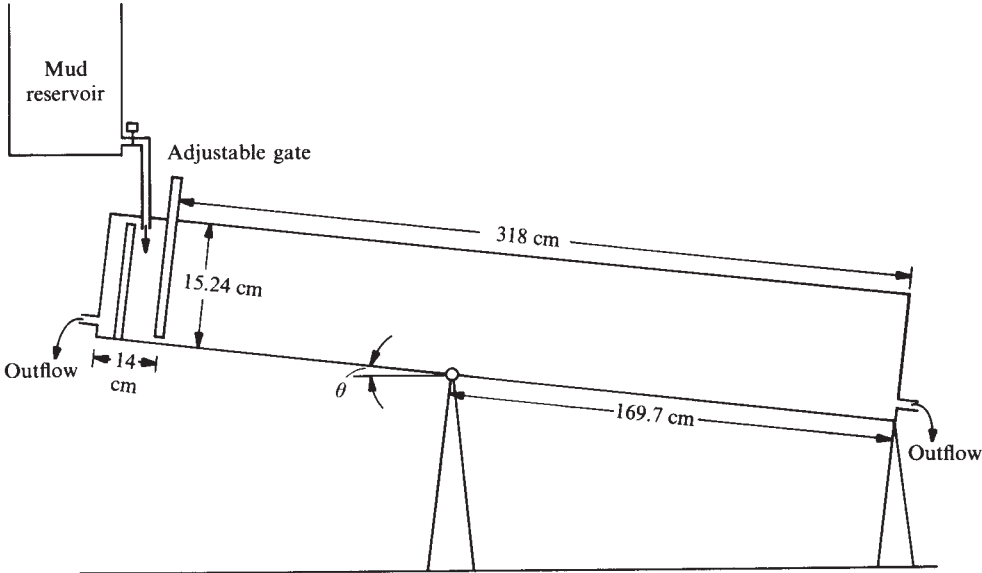


FIGURE 7. Experimental set-up for gravity currents down a dry bed.

Figure 2.3.2: Experiment setup for a mud current down an inclined plane. From Liu & Mei 1989.

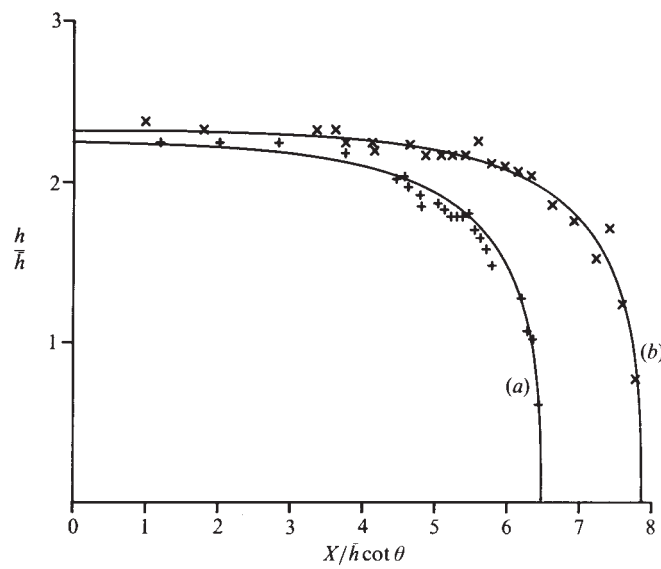


FIGURE 8. Comparison between theory and measured profiles. Curve (a) $\theta = 1.47^\circ$, phase speed = 5.22 cm/s, maximum depth = 0.71 cm and $\bar{h} = 0.31$ cm. The corresponding data points are marked +. (b) $\theta = 0.90^\circ$, phase speed = 9.46 cm/s, maximum depth = 1.22 cm and $\bar{h} = 0.51$ cm. The corresponding data points are marked x.