Notes on
1.63 Advanced Environmental Fluid Mechanics
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3.8 Oscillatory Boundary Layers

3.8.1 Stokes problem

Near the solid bottom under a wave there is a boundary layer. Let the outside flow have the tangential velocity $u = \Re U(x)e^{-i\omega t}$. Consider the ratio

$$\frac{uu_x, vu_y}{u_t} = O\left(\frac{U_o}{\omega L}\right)$$

If

$$1 \gg \frac{U}{\omega L} \gg \frac{\nu}{\omega L^2}$$

we let

$$u = u_1 + u_2 + \dots \tag{3.8.1}$$

and get from the Navier-Stokes equations the leading order approximation

$$\frac{\partial u_1}{\partial t} = \frac{\partial}{\partial t} Re\left(Ue^{-i\omega t}\right) + \nu \frac{\partial^2 u_1}{\partial y^2} \qquad y > 0 \tag{3.8.2}$$

subject to the boundary conditions that

$$u_1 \to Re\, Ue^{-i\omega t} \qquad y \to \infty \tag{3.8.3}$$

and

$$u_1 = 0 \qquad y = 0 \tag{3.8.4}$$

Let

$$u_1 = Re\left[\hat{u}_1(x,y)e^{-i\omega t} + Ue^{-i\omega t}\right] \tag{3.8.5}$$

then

$$-i\omega U - i\omega \hat{u}_1 = -i\omega U + \nu \frac{d^2\hat{u}_1}{dy^2}$$

Therefore,

$$\frac{d^2\hat{u}_1}{dy^2} + \frac{i\omega}{\nu}\hat{u}_1 = 0 \tag{3.8.6}$$

$$\hat{u}_1 \to 0, \qquad y \to \infty \tag{3.8.7}$$

$$\hat{u}_1 = -U_1(x), \qquad y = 0 \tag{3.8.8}$$

The solution is

$$\hat{u}_1 = -U(x) \exp\left[-(1-i)y\sqrt{\frac{\omega}{2\nu}}\right] \tag{3.8.9}$$

or,

$$u_1 = \Re\left\{U(x)\left[1 - \exp\left(-(1-i)y\sqrt{\frac{\omega}{2\nu}}\right)\right]e^{-i\omega t}\right\} \tag{3.8.10}$$

The sign of $\sqrt{-i}$ is chosen so that (3.8.7) is satisfied. The boundary layer thickness is

$$\delta = \sqrt{\frac{2\nu}{\omega}} \tag{3.8.11}$$

3.8.2 Induced Streaming

If the inviscid outer flow has tangential variation $\frac{dU}{dx} \neq 0$, then there is transverse flow v_1 in the boundary layer. By continuity:

$$\begin{aligned} v_1 &= -\int_0^y \frac{\partial u_1}{\partial x} dy = i e^{-i\omega t}\frac{dU}{dx}\int_0^y \left[1 - e^{-(1-i)y/\delta}\right] dy \\ &= -e^{-i\omega t}\frac{dU}{dx}\left\{y - \frac{\delta}{1-i}\left[1 - e^{-(1-i)y/\delta}\right]\right\} \end{aligned} \tag{3.8.12}$$

which is valid in $y \leq O(\delta)$ only.

Let us examine the second order:

$$\begin{aligned} \frac{\partial u_2}{\partial t} - \nu\frac{\partial^2 u_2}{\partial y^2} &= U\frac{dU}{dx} - \left(u_1\frac{\partial u_1}{\partial x} + v_1\frac{\partial u_1}{\partial y}\right) \\ &= U\frac{dU}{dx} - \left[\frac{\partial\left(u_1 u_1\right)}{\partial x} + \frac{\partial\left(u_1 v_1\right)}{\partial y}\right] \end{aligned}$$

Since U, u_1 and v_1 are simple harmonic in time, the solution for u_2 must have zeroth and second harmonics. Focussing on the zeroth harmonic by taking the average over a period

$$-\nu\frac{\partial^2 \bar{u}_2}{\partial y^2} = \overline{U\frac{dU}{dx}} - \left(\frac{\partial\overline{u_1 u_1}}{\partial x} + \frac{\partial\overline{u_1 v_1}}{\partial y}\right)$$

On the right-hand-side the last two terms $\overline{u_1 u_1}$, $\overline{u_1 v_1}$ are wave-induced Reynolds stresses. In particular $\rho\overline{u_1 u_1}$ is the rate of transporting x-momentum in the x-directionm, and $\rho\overline{u_1 v_1}$ is the rate of transporting x-momentum in y-direction.

Alternatively:

$$-\nu\frac{\partial^2 \bar{u}_2}{\partial y^2} = \frac{1}{2}\frac{\partial}{\partial x}\overline{U^2} - \frac{1}{2}\frac{\partial}{\partial x}\overline{u_1^2} - \overline{v_1\frac{\partial u_1}{\partial y}}$$

Let

$$\alpha = (1-i)/\delta \tag{3.8.13}$$

Since

$$v_1 = ie^{-i\omega t}\frac{1}{\alpha}\frac{dU}{dx}\left(\alpha y - 1 + e^{-\alpha y}\right)$$

$$\frac{\partial u_1}{\partial y} = \alpha U(x)e^{-i\omega t}e^{-\alpha y}$$

$$\overline{-v_1\,\frac{\partial u_1}{\partial y}} = \frac{1}{2}Re\left[U^*\frac{dU}{dx}\frac{\alpha^*}{\alpha}e^{-\alpha^* y}\left(\alpha y - 1 + e^{-\alpha y}\right)\right]$$

Thus

$$\begin{aligned}
-\nu\frac{\partial^2\bar{u}_2}{\partial y^2} &= G(y) \equiv \frac{1}{2}\frac{d|U|^2}{dx}\left[1-\left(1-e^{-\alpha y}\right)\left(1-e^{-\alpha^* y}\right)\right] \\
&\quad + Re\,U^*\frac{dU}{dx}\frac{\alpha *}{\alpha}e^{-\alpha^* y}\left(\alpha y - 1 + e^{-\alpha y}\right) \\
\nu\,\frac{\partial\bar{u}_2}{\partial y} &= \int_y^\infty G(y')dy' \\
\nu\,\bar{u}_2 &= \int_0^y dy''\int_{y''}^\infty G(y')dy' \\
&= -y\int_y^\infty G(y')dy' + \int_0^y y''G(y'')dy''
\end{aligned}$$

One more integration gives

$$\begin{aligned}
-\omega\bar{u}_2 &= Re\,F\,U\frac{dU^*}{dx} \\
\text{where}\quad F &= -\frac{1}{2}(1-3i)e^{-(1-i)\eta} - \frac{i}{2}e^{-(1+i)\eta} - \frac{1}{4}(1+i)e^{-2\eta} \\
&\quad + \frac{1}{2}(1+i)\eta e^{-(1-i)\eta} + \frac{3}{4}(1-i)
\end{aligned}$$

Note that as $y \to \infty$, just outside the boundary layer,

$$\bar{u}_2 = -\frac{3}{4\omega}Re\left[(1-i)\,U\frac{dU^*}{dx}\right] \tag{3.8.14}$$

By Taylor expansion we can show that for $\eta \ll 1$,

$$\bar{u}_2 \approx Re\left[(1+i)\frac{\eta}{2}U\frac{dU^*}{dx}\right] \tag{3.8.15}$$

Example : Surface gravity waves

On the free surface of water of contant depth h, let the vertical displacement be

$$\zeta = Re\left[A\left(e^{ikx} + Re^{-ikx}\right)e^{-i\omega t}\right] \tag{3.8.16}$$

where R denotes the reflection coefficient. The frequency ω frequency and and the wavenumber k are related by

$$\omega^2 = gk\tanh kh \tag{3.8.17}$$

The corresponding velocity potential is

$$\Phi = Re\left[-\frac{igA}{\omega}\frac{\cosh k(z+h)}{\cosh kh}\left(e^{ikx} + Re^{-ikx}\right)e^{-i\omega t}\right] \tag{3.8.18}$$

The inviscid horizontal velocty just above the bed boundary layer is

$$\frac{\partial}{\partial x}\Phi(x,-h,t) = \frac{gkA}{\omega\cosh kh}Re\left[\left(e^{ikx} - Re^{-ikx}\right)e^{-i\omega t}\right] \tag{3.8.19}$$

We can then identify

$$U = \frac{gkA}{\omega\cosh kh}\left(e^{ikx} - Re^{-ikx}\right) \tag{3.8.20}$$

For purely progressive waves, $R = 0$

$$U = \frac{gkA}{\omega\cosh kh}e^{ikx} \tag{3.8.21}$$

hence

$$\frac{dU^*}{dx} = -ik\frac{gkA}{\omega\cosh kh}e^{-ikx} \tag{3.8.22}$$

The induced streaming velocity is,

$$\bar{u}_2(\infty) = \frac{3}{4\omega}k\left(\frac{gkA}{\omega\cosh kh}\right)^2 \tag{3.8.23}$$

at the upper edge of the boundary layer, and

$$\bar{u}_2(\eta) \approx \frac{\eta}{2\omega}k\left(\frac{gkA}{\omega\cosh kh}\right)^2, \quad \eta \ll 1. \tag{3.8.24}$$

near the bottom of the boundary layer. The velocity profile is monotonic in height.

For purely standing waves $R = 1$, we have

$$U = \frac{gkA}{\omega\cosh kh}2i\sin kx \tag{3.8.25}$$

and

$$\frac{dU^*}{dx} = -2ik\frac{gkA}{\omega\cosh kh}\cos kx \tag{3.8.26}$$

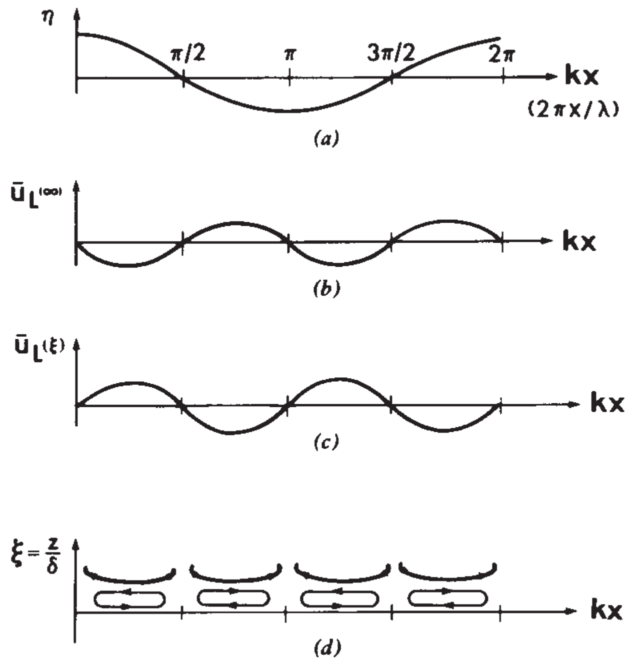

Figure 3.2 Schematic variation of mass transport velocity beneath a standing wave.

Hence

$$U\frac{dU^*}{dx} = \left(\frac{gkA}{\omega\cosh kh}\right)^2 2k\sin 2kx \tag{3.8.27}$$

It follows that

$$\bar{u}_2(\infty) = -\frac{3}{4\omega}k\left(\frac{gkA}{\omega\cosh kh}\right)^2 \sin 2kx \tag{3.8.28}$$

and

$$\bar{u}_2(\eta) \approx \frac{\eta}{2\omega}k\left(\frac{gkA}{\omega\cosh kh}\right)^2 \sin 2kx, \quad \eta \ll 1. \tag{3.8.29}$$

Thus near the bottom of the boundary layer, the streaming velocty converges toward points beneath the amplitude minima. Near the top, the opposite is true. See Figure (3.8.2).

3.8.3 Physics of the Induced Streaming

Take progressive water waves as an example: We have outside the boundary layer,

$$u_\infty = U_o\cos(\omega t - kx) \tag{3.8.30}$$

and inside the boundary layer,

$$u = U_o \left[\cos(\omega t - kx) - e^{-y/\delta} \cos(\omega t - kx - y/\delta)\right] \tag{3.8.31}$$

where the velocity amplitude U_o is related to the surface amplitude A by

$$U_o = \frac{gkA}{\omega \cosh kh} = \frac{A\omega}{\sinh kh} \tag{3.8.32}$$

Let us find the induced transverse velocity v

$$\frac{\partial u}{\partial x} = U_o \sin(\omega t - kx) - U_o e^{-y/\delta} \sin(\omega t - kx - y/\delta)$$

$$v_\infty = -\int_0^{y \gg \delta} \frac{\partial u}{\partial x} dy = -y\, U_o \sin(\omega t - kx) - \frac{1}{2} U_o k\delta \cos(\omega t - kx) + \frac{1}{2} U_o k\delta \sin(\omega t - kx)$$

Now

$$\overline{u_\infty v_\infty} = -\frac{1}{4} U_o^2 k\delta < 0$$

where the $\sin(\omega t - kx)$ terms in v_∞ are out of phase with u_∞ by $\pi/2$, hence does not contribute to the mean.

Now consider a slice of boundary layer one wavelength long. Because of periodicity, there is no net transfer of momentum or forces at two ends x_0 and $x_0 + 2\pi/k$. But the momentum transfer downwards is $\frac{U_o^2}{4} k\delta$, causing a positive shear stress. To balance it there must be a non-zero $\mu \frac{\partial \bar{u}}{\partial y}$ at all levels y below the top. Hence, the induced streaming velocity is created and $\bar{u} \neq 0$.

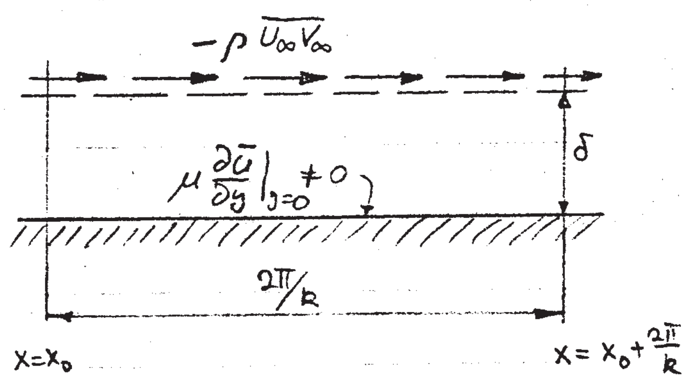

Figure 3.8.1: Reynolds stress and Induced streaming in Stokes layer

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