Notes on
1.63 Advanced Environmental Fluid Mechanics
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3-9impulsive.tex

3.9 Impulsive motion of a blunt body and tendency for separation

Ref: H. Schlichting, Boundary layer theory, p 400 ff.

As an example of unsteady boundary layer, let us consider the initial stage ($U_oT/L \ll 1$) of a boundary layer due to the impulsive start of motion near a blunt body, see the sketch in Figure 3.9.1.

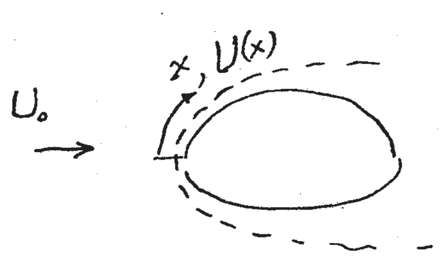

Figure 3.9.1: Boundary layer around a blunt body

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Let us start with the boundary layer approximation and introduce a perturbation expansion in powers of the small ratio U_oT/L,

$$u = u^{(1)} + \left(\frac{U_oT}{L}\right) u^{(2)} + \left(\frac{U_oT}{L}\right)^2 u^{(3)} \cdots, \tag{3.9.1}$$

$$p = p^{(1)} + \left(\frac{U_oT}{L}\right) p^{(2)} + (\frac{U_oT}{L}\Big)^2 p^{(3)} + \cdots \tag{3.9.2}$$

We then get

$$u_x^{(1)} + v_y^{(1)} + \left(\frac{U_oT}{L}\right)\left(u_x^{(2)} + v_y^{(2)}\right) + \cdots = 0, \tag{3.9.3}$$

and

$$u_t^{(1)} + \frac{U_o T}{L} u_t^{(2)} + \frac{U_o T}{L}(u^{(1)} u_x^{(1)} + v^{(1)} u_y^{(1)}) + O\left(\frac{U_o T}{L}\right)^2$$

$$= \frac{U_o T}{L} U U_x + u_{yy}^{(1)} + \frac{U_o T}{L} u_{yy}^{(2)} + O\left(\frac{U_o T}{L}\right)^2 \quad (3.9.4)$$

$$(3.9.5)$$

Equating the coefficients of $\left(\frac{U_o T}{L}\right)^0$ we get the first (leading) order perturbation equations in normalized coordinates,

$$u_x^{(1)} + v_y^{(1)} = 0, \quad (3.9.6)$$

$$u_t^{(1)} = u_{yy}^{(1)} \quad (3.9.7)$$

subject to the initial conditions:

$$u^{(1)} = v^{(1)} = 0. \quad t = 0, \quad \forall y; \quad (3.9.8)$$

and the boundary condtions

$$u^{(1)} = v^{(1)} = 0. \quad y = 0, \quad \forall t; \quad (3.9.9)$$

$$u^{(1)} = U, \quad y \to \infty \quad (3.9.10)$$

Equating the coefficient of $\left(\frac{U_o T}{L}\right)$, we get the second order perturbation equations in normalized coordinates,

$$u_x^{(2)} + v_y^{(2)} = 0, \quad (3.9.11)$$

$$u_t^{(2)} + (u^{(1)} u_x^{(1)} + v^{(1)} u_y^{(1)}) = U U_x + u_{yy}^{(2)} + O\left(\frac{U_o T}{L}\right)^2 \quad (3.9.12)$$

subject to the same initial and boundary conditions on the wall as the first order problem, except that

$$u^{(2)} \to 0, \quad y \to \infty \quad (3.9.13)$$

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To return to physical variables, we need only add the coeficient ν in front of the viscous stress term u_{yy} in (3.9.7), and (3.9.12). The first order problem for the tangential velocity is precisely the Rayleigh problem

$$u_t^{(1)} = u_{yy}^{(1)} \quad (3.9.14)$$

subject to the initial conditions:

$$u^{(1)} = 0. \quad t = 0, \quad \forall y; \quad (3.9.15)$$

and the boundary condtions

$$u^{(1)} = 0. \quad y = 0, \quad \forall t; \quad (3.9.16)$$

$$u^{(1)} = U, \quad y \to \infty \tag{3.9.17}$$

The solution is

$$u^{(1)}(x, y, t) = U(x)\mathrm{erf}(\eta) = U(x)\frac{2}{\sqrt{\pi}}\int_0^\eta e^{-\eta^2}\,d\eta \tag{3.9.18}$$

where

$$\eta = \frac{y}{\sqrt{2\nu t}} \tag{3.9.19}$$

Integrating the continuity equation (3.9.6) we get

$$v^{(1)} = -\int_0^y \frac{\partial u_1}{\partial x}dy = -\frac{dU}{dx}2\sqrt{\nu t}\int_0^\eta \mathrm{erf}(\eta)\,d\eta \tag{3.9.20}$$

To simply the notation we introduce

$$\mathrm{erf}(\eta) = \zeta_0'(\eta), \qquad \int_0^\eta \mathrm{erf}(\eta)\,d\eta = \zeta_0(\eta) \tag{3.9.21}$$

so that

$$u^{(1)} = U(x)\zeta_0'(\eta), \quad v^{(1)} = -\frac{dU}{dx}2\sqrt{\nu t}\zeta_0(\eta) \tag{3.9.22}$$

The second-order approximation is

$$u_t^{(2)} - \nu u_{yy}^{(2)} = UU_x - u^{(1)}u_x^{(1)} - v^{(1)}u_y^{(1)} \tag{3.9.23}$$

subject to the initial and boundary conditions that

$$u^{(2)}(y, 0) = 0, \quad u^{(2)}(y, t) = 0 \quad \mathrm{for} y = 0, \infty \tag{3.9.24}$$

The right hand side of (3.9.23) can be worked out so that

$$\begin{aligned} u_t^{(2)} - \nu u_{yy}^{(2)} &= UU_x\left[1 - (\mathrm{erf}(\eta))^2 + e^{-\eta^2}\int_0^\eta \mathrm{erf}(\eta)\,d\eta\right] \\ &= UU_x\left[1 - (h')^2 + hh''\right] = UU_xF(\eta) \end{aligned} \tag{3.9.25}$$

A similarity solution is possible. Let us seek a one-parameter transformation,

$$u^{(2)} = \lambda^a u^{(2)\prime}, \quad t = \lambda^b t', \quad y = \lambda^c y'$$

From (3.9.23) we get

$$\lambda^{a-b}\frac{\partial u^{(2)\prime}}{\partial t'} - \nu\lambda^{a-2c}\frac{\partial^2 u^{(2)\prime}}{\partial y'^2} = UU_xF(\lambda^{c-b/2}\eta')$$

Note that x is just a parameter. Clearly $a = b = 2c$ so that we can take

$$\frac{u^{(2)}}{t} = f(\eta)UU_x \tag{3.9.26}$$

Substituting (3.9.26) into (3.9.25), we get a linear ordinary differential equation

$$f'' + 2\eta f' - 4f = 4\left[(\zeta_0')^2 - \zeta_0\zeta_0'' - 1\right] \tag{3.9.27}$$

subject to the boundary conditions that

$$f = 0, \quad \eta = 0, \infty \tag{3.9.28}$$

The solution is not difficult (see Schlichting, eq. 15.43, p. 400).

$$\begin{aligned} f &= \operatorname{erfc}(\eta)\left[-\frac{3}{\sqrt{\pi}}e^{-\eta^2} + 2 - \left(\frac{3}{\sqrt{\pi}} + \frac{4}{3\pi\sqrt{\pi}}\right) + \frac{\sqrt{\pi}}{2}(2\eta^2+1)\right] \\ &+ \frac{1}{2}(2\eta^2-1)\operatorname{erfc}^2(\eta) + \frac{2}{3}e^{-2\eta^2} \\ &+ e^{-\eta^2}\left[\frac{\eta}{\sqrt{\pi}} - \frac{4}{3\pi} + \eta\left(\frac{3}{\sqrt{\pi}} + \frac{4}{3\pi\sqrt{\pi}}\right)\right] \end{aligned} \tag{3.9.29}$$

The solution is plotted in Figure 3.9.2.

The total solution is

$$u = U\operatorname{erf}(\eta) + tUU_x f(\eta) \tag{3.9.30}$$

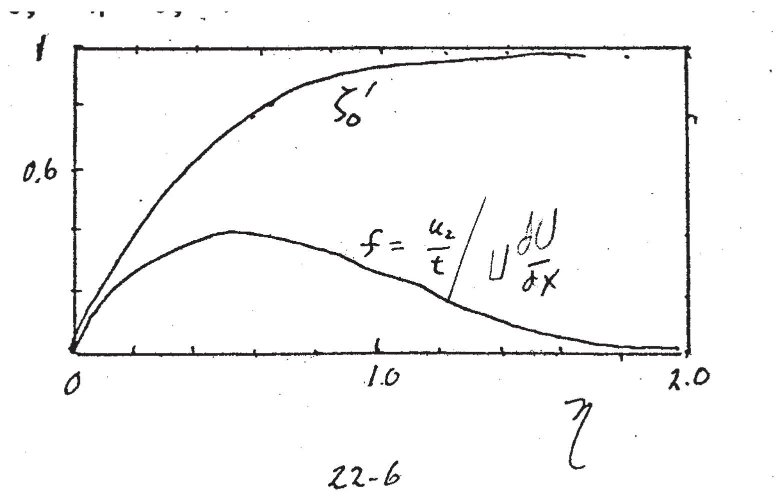

Figure 3.9.2: Solution to the problem of impulsive start.

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Separation

For a given $U(x)$ when and where will separation first occur? Namely, when is

$$\frac{\partial u}{\partial y} = 0 at y = 0$$

Let us use (3.9.30) for a crude estimate. Since

$$\frac{\partial u}{\partial y} = [U(\text{erf}\eta)' + UU_x\, t\, f'(\eta)]\,\frac{\partial \eta}{\partial y}$$

It can be show n that at $\eta = 0$,

$$(\text{erf}\eta)' = \frac{2}{\sqrt{\pi}}, \quad f'(\eta) = \frac{2}{\sqrt{\pi}}\left(1 + \frac{4}{3\pi}\right)$$

It follows that

$$U + t_s\left(1 + \frac{4}{3\pi}\right)UU_x = 0$$

or

$$t_s = -\frac{0.7}{UU_x} \tag{3.9.31}$$

Note that $t_s > 0$ only for $U_x < 0$, i.e., a decelerated flow. This is a very crude and mathematically illigitimate estimate since we are equating two terms of different order.

Neveltherless let us apply this result to the impulsive flow passing a circular cylinder from the left. Let U_o be the constant velocity at infinity and the polar angle θ be measured from the upstream stagnation point, then $x = a\theta$ where a is the radius, see Figure 3.9.3. It is well known in the potential theory that the potential is

$$\phi = U_o\left(r + \frac{a^2}{r}\right)\cos(\pi - \theta)$$

The tangential velocity along the cylinder $r = a$ is

$$\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{U_o}{r}\left(r + \frac{a^2}{r}\right)\sin(\pi - \theta), \quad r = a$$

or

$$U = 2U_o\sin(\pi - \theta) = 2U_o\sin(\theta) = 2U_o\sin x/a$$

The minimum t_s occurs at the riear stagnation point, $x = \pi a$ at which

$$t_s = \frac{0.35a}{U_o}, \quad \text{or} \quad \frac{U_o t_s}{a} = 0.35$$

Note that the last condition indicates the illigitimacy of this estimate. Nevertheless we use it here as an order-of-magnitude guide which may be improved by working out higher order terms.

In offshore stuctures, wave induced oscillatory flows acound a pile can be separated and hence affect the drag force on the pile. As an order estimate we take $U_o = \omega A$ where ω =frequency and A =wave amplitude. Hence there is no separation if

$$\frac{\omega A t_s}{a} < 0.35, \quad \text{or} \quad \frac{A}{a} < \frac{0.35}{\omega t_s}$$

Since flow changes direction after every half period π/ω, there is no separation in every half period if

$$\frac{A}{a} < \frac{0.35}{\pi} = 0.1$$

This is of course very crude. Experimentally Keulegan and Carpenter have estiblished that separation occurs in waves if A/a exceeds 1. The ratio A/a is now known as the Keulegan and Carpenter number.

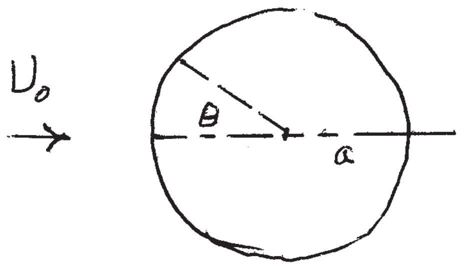

Figure 3.9.3: Definition of coordinates for a circular cylinder.

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