Lecture notes in Fluid Dynamics
(1.63J/2.01J)
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4-6selw-therm.tex

4.6 Selective withdrawl of thermally stratified fluid

[References]:

R.C. Y. Koh, 1966 *J. Fluid Mechanics*, **24**, pp. 555-575.
Brooks, N. H., & Koh, R. C. Y., Selective withdrawal from density stratified reservoirs. *J Hydraulics,* ASCE, HY4, July 1969. 1369-1400.
Ivey, G. N.
Monosmith, et. al.

We now extend the analysis in Chapter II on isothemal withdrawl and consider the slow and steady flow of a thermally stratified fluid into a two- dimensional line sink.

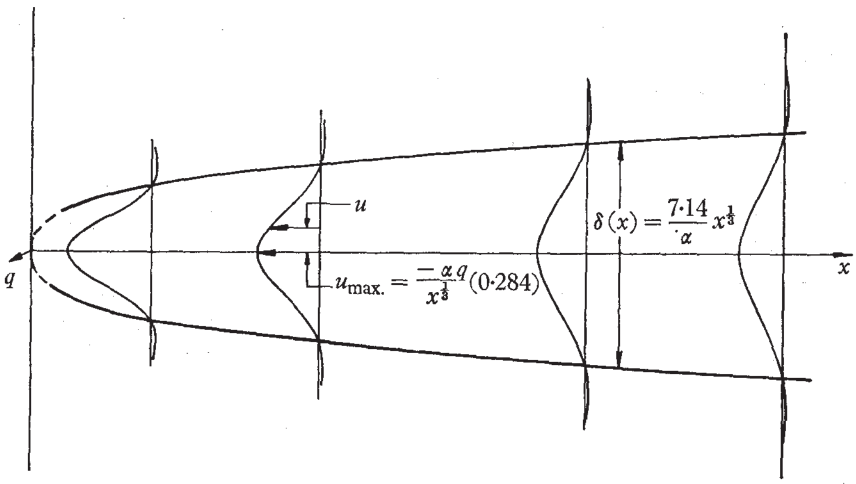

FIGURE 3. Viscous stratified flow towards a line sink: the withdrawal layer.

Figure 4.6.1: Sketch of velocity profiles across the layer draining int to a line sink, from Koh, 1966.

Thermal diffusion and convection now comes into play.

4.6.1 Governing equations

We begin with the general law of mass conservation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = \frac{\partial \rho}{\partial t} + \vec{q} \cdot \nabla \rho + \rho \nabla \cdot \vec{q} = 0 \tag{4.6.1}$$

In environmental problems the range of temperature variation is within a few tens of degrees. The fluid density varies very little and obeys the following equation of state

$$\rho = \rho_o \left[1 - \beta (T - T_o)\right] \tag{4.6.2}$$

where T denotes the temperature and β the coefficient of thermal expansion which is usually very small. Hence

$$\frac{\vec{q} \cdot \nabla \rho}{\rho \nabla \cdot \vec{q}} = O\left(\frac{\Delta \rho}{\rho}\right) \ll 1$$

and

$$\frac{\frac{1}{\rho}\frac{\partial \rho}{\partial t}}{\nabla \vec{u}} \sim \frac{\Delta \rho}{\rho} \ll 1$$

It follows that (4.6.1) is well approximated by

$$\nabla \cdot \vec{q} = 0 \tag{4.6.3}$$

which means that water is essentially incompressible. In two dimensions, we have

$$u_x + w_z = 0 \tag{4.6.4}$$

Next, energy conservation requires that

$$\frac{\partial T}{\partial t} + \vec{q} \cdot \nabla T = D \nabla^2 T \tag{4.6.5}$$

Let

$$T = \overline{T} + T' \tag{4.6.6}$$

where $\overline{T}$ represents the static temperature when there is no motion, and T' the motion-induced temperature variation. Therefore,

$$T - T_o = \left(\overline{T}(z) - T_o\right) + T'(x, z, t) \tag{4.6.7}$$

and

$$\frac{\partial T'}{\partial t} + \vec{q} \cdot \nabla \bar{T} + \vec{q} \cdot \nabla T' = D \nabla^2 \overline{T} + D \nabla^2 T' \tag{4.6.8}$$

The static temperature must satisfy

$$\nabla^2 \overline{T} = 0 \tag{4.6.9}$$

In a large lake with depth much smaller than the horizontal extent, the static temperature is essentially uniform horizontally. The Laplace equation reduces to

$$D\frac{d^2\overline{T}}{dz^2} = 0, \quad \text{implying} \quad \frac{d\overline{T}}{dz} = \text{constant} \tag{4.6.10}$$

The dynamic part is then gorvened by

$$\frac{\partial T'}{\partial t} + u\frac{\partial T'}{\partial x} + w\frac{\partial T'}{\partial z} + w\frac{\partial \bar{T}}{\partial z} = D\nabla^2 T' \tag{4.6.11}$$

The exact equations for momentum balance are, in two dimensions,

$$\rho\left(\frac{\partial u}{\partial t} + \vec{q}\cdot\nabla u\right) = -\frac{\partial p}{\partial x} + \mu\nabla^2 u \tag{4.6.12}$$

$$\rho\left(\frac{\partial w}{\partial t} + \vec{q}\cdot\nabla w\right) = -\frac{\partial p}{\partial z} - \frac{\partial \overline{p}}{\partial z} - g\rho_o\left[1 - \beta(\overline{T} + T' - T_o\right] + \mu\nabla^2 w \tag{4.6.13}$$

where $\overline{p}$ denotes the static part, which must satisfy

$$0 = -\frac{\partial \overline{p}}{\partial z} - g\rho_o\left[1 - \beta(\overline{T} - T_o)\right] \tag{4.6.14}$$

Taking the differenece of the two preceding equations, we find the equation for the dynamic part

$$\rho\left(\frac{\partial w}{\partial t} + \vec{q}\cdot\nabla w\right) = -\frac{\partial p}{\partial z} + g\rho_o\beta T' + \mu\nabla^2 w \tag{4.6.15}$$

4.6.2 Approximation for slow and steady flow

For sufficiently slow flows, inertia terms can be ignored. Expecting that vertical motion is suppressed, we further assume that the vertical length scale δ is much smaller than the horizontal scale L, so that $\partial/\partial x \ll \partial/\partial z$. The 2-D momentum equations can then be simplified to

$$0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial z^2} \tag{4.6.16}$$

$$0 = -\frac{\partial p}{\partial z} + g\beta\rho_o T' + \mu\frac{\partial^2 w}{\partial z^2} \tag{4.6.17}$$

Similarly we can linearize (4.6.11) to get

$$w\frac{d\bar{T}}{dz} = D\frac{\partial^2 T'}{\partial z^2} \tag{4.6.18}$$

Together (4.6.4), (4.6.18), (4.6.16) and (4.6.17) complete the lineaized governing equations.

Eliminating p from (4.6.16) and (4.6.17), we get

$$\mu \frac{\partial^2}{\partial z^2}(u_z - w_x) = g\beta\rho_o \frac{\partial T'}{\partial x} \tag{4.6.19}$$

Since

$$\frac{w}{u} = O\left(\frac{\delta}{L}\right) \ll 1, \quad \frac{w_x}{u_z} = O\left(\frac{\delta}{L}\right)^2 \ll 1$$

we can omit the second term on the left of (4.6.19). In terms of the stream function defined by

$$u = \psi_z, \quad w = -\psi_x \tag{4.6.20}$$

(4.6.19) becomes

$$\boxed{\frac{\partial^4 \psi}{\partial z^4} = \frac{g\beta\rho_o}{\mu}\frac{\partial T'}{\partial x}} \tag{4.6.21}$$

Equation (4.6.18) can be written as

$$\boxed{\psi_x \frac{d\overline{T}}{dz} = D\frac{\partial^2 T'}{\partial z^2}} \tag{4.6.22}$$

We now have just two equations for two unknowns ψ and T'. The boundary conditions are

$$T'u, w \downarrow 0, \quad \text{as} \quad z \uparrow \pm\infty \tag{4.6.23}$$

or

$$\psi, \psi_z T \downarrow 0, \quad \text{as} \quad z \uparrow \pm\infty. \tag{4.6.24}$$

Let the volume rate of withdrawal be prescribed, we must then require the integal condition:

$$\int_{-\infty}^{\infty} u\,dz = -q, \qquad \text{implying} \quad \psi(x, z = \infty) - \psi(x, z = -\infty) = q. \tag{4.6.25}$$

4.6.3 Normalization

Let

$$\psi = q\psi^*, \quad T = T_o T^*, \quad x = Lx^*, \quad z = \delta z^* \tag{4.6.26}$$

Physically it is natural to choose the characteristic depth of thermal gradient as the global length scale L:

$$L = -\left(\beta\frac{d\bar{T}}{dz}\right)^{-1} \tag{4.6.27}$$

The scales T_o and δ are yet to be specified.

The dimensionless (4.6.21) reads

$$\frac{q}{\delta^4}\left(\frac{\partial^4\psi}{\partial z^4}\right)^* = \frac{g\beta\rho_o T_o}{\mu L}\left(\frac{\partial T'}{\partial x}\right)^*,$$

hence we choose

$$\frac{q}{\delta^4} = \frac{g\beta\rho_o T_o}{\mu L} \tag{4.6.28}$$

so that

$$\boxed{\left(\frac{\partial^4\psi}{\partial z^4}\right)^* = \left(\frac{\partial T'}{\partial x}\right)^*} \tag{4.6.29}$$

Similarly, (4.6.22) becomes

$$\frac{T_o}{\delta^2}\left(\frac{\partial^2 T'}{\partial z^2}\right)^* + \frac{q}{DL}\frac{d\bar{T}}{dz}\left(\frac{\partial\psi}{\partial x}\right)^* = 0$$

after normalization, suggesting the choice of

$$\frac{T_o}{\delta^2} = \frac{q\, d\overline{T}/dz}{DL} \tag{4.6.30}$$

so that

$$\boxed{\left(\frac{\partial^2 T}{\partial z^2}\right)^* + \left(\frac{\partial\psi}{\partial x}\right)^* = 0} \tag{4.6.31}$$

Eqs. (4.6.28) and (4.6.30) can be solved to give the scales

$$\delta = \frac{L^{1/3}}{\alpha}, \quad \text{where} \quad \alpha = \frac{g\beta\rho_o}{d\mu D}\frac{d\bar{T}}{dz} \tag{4.6.32}$$

and

$$T_o = q\left(\frac{\delta^2\, d\bar{T}/dz}{DL}\right) \tag{4.6.33}$$

The flux condition is normalized to

$$\psi^*(\infty) - \psi^*(-\infty) = 1 \tag{4.6.34}$$

4.6.4 Similarity solution

Let us try a one-parameter *similarity* transformation

$$x = \lambda^a \hat{x}, \qquad z = \lambda^b \hat{z}, \qquad \psi = \lambda^c \hat{\psi}, \qquad T' = \lambda^d \hat{T} \tag{4.6.35}$$

The exponents a, b, c and d will be chosen so that the boundary value problem is formally the same as the original one To achieve invariance of (4.6.34), we set $c = 0$. In addition we set

$$\lambda^{-4b} = \lambda^{d-a}$$

for (4.6.29), and

$$\lambda^{d-2b} = \lambda^{-a}$$

for (4.6.31). Hence, $d - a = -4b$ and $a - 2b = -d$ implying

$$b = -d, \quad a = 3b = -3d. \tag{4.6.36}$$

These relationships among the exponents suggest the following new similarity variables:

$$\psi = f(\zeta), \quad T = \frac{h(\zeta)}{x^{1/3}} \tag{4.6.37}$$

with

$$\zeta = \frac{z}{x^{1/3}} \tag{4.6.38}$$

It is easily verified that these variables are invareiant under the similarity transformation. Carrying out the differentiations

$$\begin{aligned}
\psi_z &= \frac{f'}{x^{1/3}}, \psi_{zzzz} = \frac{f''''}{x^{4/3}} \\
T_x &= h' \frac{1}{x^{1/3}} \left(-\frac{1}{3} \frac{z}{x^{4/3}} \right) + h \left(-\frac{1}{3} \right) \frac{1}{x^{4/3}} \\
&= -\left(\frac{1}{3} \zeta h' \frac{1}{x^{4/3}} + \frac{h}{3} \frac{1}{x^{4/3}} \right)
\end{aligned}$$

we get from (4.6.29)

$$\boxed{f'''' = -\frac{1}{3}(\zeta h' + h)} \tag{4.6.39}$$

Since

$$\begin{aligned}
T_z &= h' \frac{1}{x^{2/3}}, \quad T_{zz} = h'' \frac{1}{x} \\
\psi_x &= f' \frac{z}{x^{4/3}} \left(-\frac{1}{3} \right) = -\frac{1}{3} f' \zeta \frac{1}{x}
\end{aligned}$$

we get from (4.6.31)

$$h''\frac{1}{x} - \frac{f'\zeta}{3}\frac{1}{x} = 0$$

or

$$\boxed{h'' - \frac{\zeta}{3}f' = 0} \tag{4.6.40}$$

The boundary conditions are transformed to

$$f(\infty) - f(-\infty) = -1 \tag{4.6.41}$$

and

$$f, f', h \downarrow 0 \quad \text{as} \quad \zeta \to \pm\infty \tag{4.6.42}$$

Mathematically, the similarity transformation has enabled us to reduce the boundary value problem involving partial differential equations to one with ordinary differential equations (4.6.39), (4.6.40), (4.6.41), and (4.6.42). As long as x and z lie on the parabola $z = \text{const}\, x^{1/3}$, ψ^* and $T^* x^{*1/3}$ are the same. From the transformation, we can also deduce that the boundary of the zone affected by the flow is a parabola,

$$\delta \sim x^{1/3} \tag{4.6.43}$$

Along the centerline $z = \zeta = 0$, the velocity varies as

$$U_{\text{max}} \sim \psi_x \sim x^{-1/3} \tag{4.6.44}$$

and the temperature varies as

$$T_{\text{max}} \sim x^{1/3} \tag{4.6.45}$$

The boundary value problem can now be solved by numerical means (such as Runge-Kutta). Numerical results by Koh (1966, Fig. 4) are shown in Figure (4.6.2).

In Koh (1966), stratification in fluid density is associated with the variation of concentration of a diffusive substance instead of temperature. The fluid density is governed by a diffusion equation formally the same as that for temperature here. To use his numerical results, f_0, h_0 in his plots are replaced by our $f, -h$ shown here. Extensive discussion on experimental confirmation as well as the three dimensional theory for a point sink can be found in Koh.

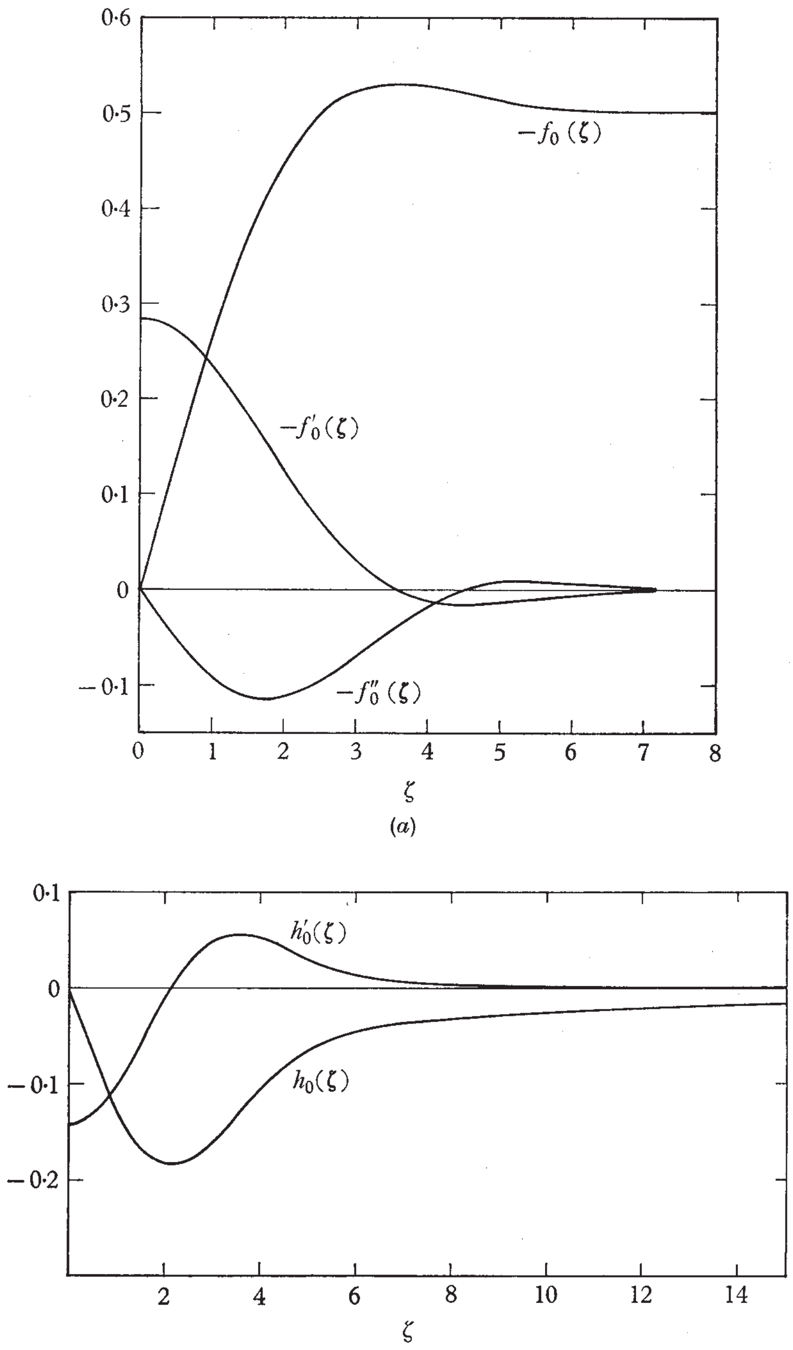

FIGURE 4. (*a*) The non-dimensional stream function and its derivatives for the two-dimensional case. (*b*) The non-dimensional density function and its first derivative for the two-dimensional case.

Figure 4.6.2: Temperature and velocity profiles across the layer draining into a line sink, from Koh, 1966.