Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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CHAPTER 5. RUDIMENTS OF HYDRODYNAMIC INSTABILITY

References:
Drazin: **Introduction to Hydrodynamic Stability**
Chandrasekar: **Hydrodynamic and Hydromagnatic Instability**
Stuart: **Hydrodynamic Stability**, in Rosenhead (ed) : **Laminar boundary layers**
Lin : **Theory of Hydrodynamic Stability**
Drazin and Reid : **Hydrodynamic Stability**
Schlichting: **Boundary Layer Theory**
C.S. Yih, 1965, **Dynamics of Inhomogeneous Fluids**, MacMillan.
Kundu : **Fluid Mechanics**.

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Various factors can be crucial to hydrodynamic instability.: shear, gravity, surface tension, heat, centrifugal force, etc. Some instabilities lead to a different flow; some to turbulence.

We only discuss the linearized analysis of instability of a few sample problems. More examples will be discussed in later chapters.

5.1 Kelvin-Helmholz Instability for continuous shear and stratification

5.1.1 Heuristic reasoning

Due to viscosity, shear flow exists along the boundary of a jet, a wake or a plume . On the interface of salt and fresh water, density stratification further comes into play. When will dynamic instability occur?

Refering to Figure 5.1.1, Consider two fluid parcels, each of unit volume, at levels z and $z+dz$. Let their positions be interchanged. To overcome gravity, the force needed to lift the heavier fluid parcel by η is

$$g\left[\overline{\rho}(z)-\overline{\rho}(z+\eta)\right]=-g\frac{d\overline{\rho}}{dz}\eta.$$

Work needed to lift the heavier parcel by dz is

$$-g\frac{d\overline{\rho}}{dz}\int_0^{dz}\eta d\eta=-\frac{1}{2}d\overline{\rho}\,dz.$$

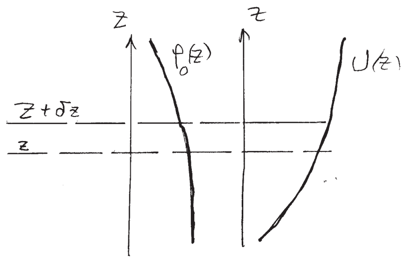

Figure 5.1.1: Exchanging fluid parcels in a stratified shear flow

Similarly, the work needed to push the light parcel down by dz is $-\frac{1}{2}gd\overline{\rho}dz$. Therefore the total work needed is

$$-gd\overline{\rho}\,dz.$$

Before the exchange, the total kinetic energy is

$$\frac{1}{2}\overline{\rho}[U^2 + (U + dU)^2]$$

where Boussinesq approximation is used. After the exchange, the parcels mix with the surrounding fluid and attain the average velocity

$$(U + U + dU)/2 = U + dU/2$$

Therefore the total kinetic energy is

$$\overline{\rho}(U + dU/2)^2$$

The available kinetic energy is the difference between the kinetic energies before and after the exchange.

$$\frac{\overline{\rho}}{2}\left\{U^2 + (U + dU)^2 - 2(U + dU/2)^2\right\} = \frac{\overline{\rho}}{4}dU^2.$$

If the net available kinetic energy exceeds the work needed for the exchange, the disturbance will grow and the flow will become unstable, i.e.,

$$\frac{\overline{\rho}dU^2}{4} > -gd\overline{\rho}dz$$

Let the Richardson number be defined by

$$R_i \equiv \frac{-\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}}{\left(\frac{dU}{dz}\right)^2} \tag{5.1.1}$$

Instabilty occurs if

$$\frac{1}{4} > R_i \equiv \frac{-\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}}{\left(\frac{dU}{dz}\right)^2} \tag{5.1.2}$$

(Chandrasekar, 1961).

Remark: A slightly more accurate estimate can be made without Boussinesq approximation. Before the exchange, the total kinetic energy is

$$\frac{1}{2}\left\{\overline{\rho}U^2 + (\overline{\rho} + d\overline{\rho})(U + dU)^2\right\}.$$

After the exchange, the parcels mix with the surrounding fluid and attain the average velocity

$$(U + U + dU)/2 = U + dU/2$$

but their densities are preserved. Therefore the total kinetic energy is

$$\frac{1}{2}(\overline{\rho} + \overline{\rho} + d\overline{\rho})(U + dU/2)^2$$

The available kinetic energy is the difference between the kinetic energies before and after the exchange.

$$\frac{\overline{\rho}}{4}dU^2 - UdUd\overline{\rho} + \frac{1}{4}d\overline{\rho}dU^2$$

Ignoring the last term, the necessary condition for instability is

$$\frac{\overline{\rho}}{4}dU^2 - UdUd\overline{\rho} + \frac{1}{4}d\overline{\rho}dU^2 > -gd\overline{\rho}dz$$

or

$$\frac{1}{4} - \frac{\frac{1}{\overline{\rho}}\frac{d\overline{\rho}}{dz}}{\frac{1}{U}\frac{dU}{dz}} + \frac{1}{4}\frac{d\overline{\rho}}{\overline{\rho}} > \frac{-\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}}{\left(\frac{dU}{dz}\right)^2}$$

On the left-hand side, the third term is negligible compared to the first. The ratioi of the second term on the left to the term on the right is

$$\frac{U}{g}\frac{dU}{dz} \sim \frac{U^2}{gL}$$

where L is the length scale of stratification. As long as the last ratio is very small, the criterion $R_i < 1/4$ still holds.

Let us confirm the heuristic result but the linearize theory.

5.1.2 Linearized instability theory for continuous shear and stratification.

Let the total flow field be $(U+u, w, P+p, \bar{\rho}+\tilde{\rho})$ where $U, P, \bar{\rho}$ represent the backgraound flow $(u, w, p, \tilde{\rho})$ the dynamical perturbations of infinitesimal magnitude. The linearized governing equations are: continuity:

$$u_x + w_z = 0 \tag{5.1.3}$$

incompressiblity:

$$\tilde{\rho}_t + U\tilde{\rho}_x + w\overline{\rho}' = 0 \tag{5.1.4}$$

where

$$\overline{\rho}' \equiv \frac{d\overline{\rho}}{dz}$$

and momentum conservation:

$$\overline{\rho}\left(u_t + Uu_x + wU_z\right) = -p_x \tag{5.1.5}$$

$$\overline{\rho}\left(w_t + Uw_x\right) = -p_z - \tilde{\rho}g. \tag{5.1.6}$$

where $\tilde{\rho}$ denotes the perturbation of density from $\bar{\rho}$.

Let us follow Miles and introduce a new unknown η by enoting $\tilde{\rho} = -\overline{\rho}'\eta$, then Eqn. (5.1.4) gives

$$\eta_t + U\eta_x = w \tag{5.1.7}$$

Consider

$$\eta = F(z)e^{ik(x-ct)}, \tag{5.1.8}$$

where

$$c = \omega/k = c_r + ic_i.$$

For fixed k the flow is unstable if $c_i > 0$, since

$$e^{-ikct} = e^{-ikc_rt}e^{kc_it}.$$

Let

$$\{u, w, p, \tilde{\rho}\} = \{\hat{u}(z), \hat{w}(z), \hat{p}(z), -\overline{\rho}'F(z)\}\, e^{ik(x-ct)} \tag{5.1.9}$$

We get from Eqn. (5.1.7)

$$\hat{w} = ik(U-c)F,, \tag{5.1.10}$$

from Eqn. (5.1.3)

$$\hat{u} = -[(U-c)F]', \tag{5.1.11}$$

and from Eqn. (5.1.5)

$$\overline{\rho}\left(ik(U-c)\hat{u} + U'[ik(U-c)F]\right) = -ik\hat{p}$$

or, after mlultiplying by ik, and using (5.1.11),

$$\overline{\rho}[(U-c)(-)[(U-c)F]' + U'(U-c)F] = -\hat{p},$$

hence

$$\hat{p} = \overline{\rho}(U-c)^2 F'. \tag{5.1.12}$$

Substituting Eqns. (5.1.9), (5.1.10), (5.1.11) and (5.1.12) into Eqn. (5.1.6), we get

$$\left[\overline{\rho}(U-c)^2 F'\right]' + \overline{\rho}\left[N^2 - k^2(U-c)^2\right] F = 0, \tag{5.1.13}$$

where N is the Brunt-Väisälä frequency defined by

$$N^2 = -\frac{g}{\overline{\rho}}\frac{d\overline{\rho}}{dz}. \tag{5.1.14}$$

Let the top and bottom be rigid walls, then $w = 0$. Hence,

$$\eta = 0 \;\text{ i.e., }\; F = 0, \qquad z = 0, d. \tag{5.1.15}$$

The argument is unchanged if the top and bottom are at $z = \infty$ and $z = -\infty$. Equations (5.1.13) and (5.1.15) consititute an eigenvalue problem where $c = c_r + ic_i$ is the eigenvalue. If $c_i > 0$, instability occurs.

5.1.3 A necessary condition for instability (J.W. Miles, L. N. Howard).

For brevity we set $W = U - c$. Miles further introduce $G = \sqrt{W}F$, so that Eqn. (5.1.13) becomes

$$(\overline{\rho}WG')' - \left[\frac{1}{2}(\overline{\rho}U')' + k^2\overline{\rho}W + \frac{\overline{\rho}}{W}\left(\frac{1}{4}U'^2 - N^2\right)\right] G = 0. \tag{5.1.16}$$

The boundary conditions are

$$G(0) = G(d) = 0. \tag{5.1.17}$$

Multiplying Eqn. (5.1.16) by G^* and integrating by parts

$$\int_0^d \left\{\overline{\rho}W\left(\mid G' \mid^2 + k^2 \mid G \mid^2\right) + \frac{1}{2}(\overline{\rho}U')'|G|^2 + \overline{\rho}\left(\frac{1}{4}U'^2 - N^2\right) W^* \mid \frac{G}{W} \mid^2\right\} dz = 0. \tag{5.1.18}$$

We now seek the necessary condition for instability, i.e., $c_i \neq 0$. Writing

$$W = (U - c_r) - ic_i \qquad W^* = (U - c_r) + ic_i$$

and substituting these in (5.1.18), we get

$$\int_0^d \left\{\overline{\rho}(U - c_r - ic_i)\left(\mid G' \mid^2 + k^2 \mid G \mid^2\right)\right.$$

$$+\frac{1}{2}\left(\overline{\rho}U'\right)'|G|^2+\overline{\rho}\left(\frac{1}{4}U'^2-N^2\right)(U-c_r+ic_i)\mid\frac{G}{W}\mid^2\Big\}\,dz=0.$$

Separating the imaginary part, we get, if $c_i \neq 0$,

$$\int_0^d \overline{\rho}\left((\mid G'\mid^2+k^2\mid G\mid^2\right)dz+\int_0^d \overline{\rho}\left(N^2-\frac{1}{4}(U')^2\right)\mid\frac{G}{W}\mid^2 dz=0.$$

For this to be true it is necessary that $N^2 < \frac{1}{4}(U')^2$ or

$$R_i=\frac{N^2}{(U')^2}=\frac{-\frac{g}{\bar{\rho}}\frac{d\bar{\rho}}{dz}}{\left(\frac{dU}{dz}\right)^2}<\frac{1}{4}. \tag{5.1.19}$$

This confirms the heurisic result as the necessary (but not sufficient) condition for instabililty (J.W. Miles, L. N. Howard).

Remark on Brunt-Väisälä frequency:

Brunt-Väisälä frequency is a parmeter characteristic of a stratified fluid. It is the natural frequency of oscillations of a fluid parcel displaced slightly from equilibrium. Let a fluid parcel of unit volume be moved up by a small distance η. The parcel is subject to a downward gravitational force equal in magnitude to $-g\overline{\rho}'\eta$ (i.e., the upward force $g\overline{\rho}'\eta$ needed to lift it up), while the inertial force is $\rho d^2\eta/dt^2$. Newton's law requires

$$\bar{\rho}\frac{d^2\eta}{dt^2}=g\bar{\rho}'\eta, \quad \text{or} \quad \bar{\rho}\frac{d^2\eta}{dt^2}-g\bar{\rho}'\eta=0. \tag{5.1.20}$$

Hence the natural frequency is the Brunt-Väisälä frequency:

$$N=\sqrt{\frac{-g\bar{\rho}'}{\bar{\rho}}} \tag{5.1.21}$$