

Lecture Notes on Fluid Dynamics

(1.63J/2.21J)

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5.2 Kelvin-Helmholtz instability of flow with discontinuous shear and stratification.

5.2.1 Nonlinear formulation

Consider two immiscible fluids. The upper ($z > 0$) and lower ($z < 0$) fluids have different densities ρ_1 and ρ_2 . Before any perturbations appear the lower fluid is stationary, while the upper fluid moves at the steady and uniform velocity $\vec{q}_o = U\vec{i}$. With gravity, this is called the Kelvin-Helmholtz instability problem. For the time being, let us assume that there is no gravity.

Now let there be a disturbance on the interface whose vertical displacement is $z = \zeta(x, t)$. The corresponding disturbances of velocity and pressure are \vec{q}_1 and p_1 in the upper fluid and \vec{q}_2 and p_2 in the lower fluid.

Let us first formulate the nonlinear problem. In the upper fluid, we have

$$\vec{q}_1 = \nabla\Phi_1, \quad \text{and} \quad \nabla^2\Phi_1 = 0, \quad z > 0 \quad (5.2.1)$$

and the Bernoulli equation,

$$\frac{\partial\Phi_1}{\partial t} + \frac{(\nabla\Phi_1)^2}{2} = -\frac{p_1}{\rho_1} + C_1 \quad (5.2.2)$$

In the lower fluid, we have :

$$\vec{q}_2 = \nabla\Phi_2, \quad \text{and} \quad \nabla^2\Phi_2 = 0 \quad z < 0 \quad (5.2.3)$$

with the Bernoulli equation,

$$\frac{\partial\Phi_2}{\partial t} + \frac{(\nabla\Phi_2)^2}{2} = -\frac{p_2}{\rho_2} + C_2 \quad (5.2.4)$$

On the interface $F = z - \zeta(x, t) = 0$ the kinematic conditions are

$$\frac{\partial F}{\partial t} + \nabla\Phi_1 \cdot \nabla F = 0 \quad (5.2.5)$$

and

$$\frac{\partial F}{\partial t} + \nabla\Phi_2 \cdot \nabla F = 0 \quad (5.2.6)$$

or, more explicitly,

$$\frac{\partial\zeta}{\partial t} + \frac{\partial\Phi_1}{\partial x} \frac{\partial\zeta}{\partial x} = \frac{\partial\Phi_1}{\partial z}, \quad z = \zeta, \quad (5.2.7)$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \Phi_2}{\partial x} \frac{\partial \zeta}{\partial x} = \frac{\partial \Phi_2}{\partial z}, \quad z = \zeta, \quad (5.2.8)$$

The dynamic condition is

$$p_1 = p_2, \quad \text{or } z = \zeta. \quad (5.2.9)$$

$$\rho_1 \left(\frac{\partial \Phi_1}{\partial t} + \frac{(\nabla \Phi_1)^2}{2} - C_1 \right) = \rho_2 \left(\frac{\partial \Phi_2}{\partial t} + \frac{(\nabla \Phi_2)^2}{2} - C_2 \right), \quad \text{or } z = \zeta. \quad (5.2.10)$$

Far away from the interface,

$$\nabla \Phi_1 \rightarrow U_1 \vec{i}, \quad z \rightarrow \infty, \quad (5.2.11)$$

$$\nabla \Phi_2 \rightarrow U_2 \vec{i}, \quad z \rightarrow -\infty. \quad (5.2.12)$$

Note that in the absence of any transient disturbances, $\nabla \Phi_1 = U_1 \vec{i}$, $\nabla \Phi_2 = U_2 \vec{i}$, so that

$$\rho_1 \left(C_1 - \frac{U_1^2}{2} \right) = \rho_2 \left(C_2 - \frac{U_2^2}{2} \right) \quad (5.2.13)$$

5.2.2 Linearization

We now limit ourselves to infinitesimal disturbances. Specifically, the interface displacement is infinitesimal when compared to other length scales.

$$\Phi_1 = U_1 x + \phi_1, \quad \text{and } \Phi_2 = U_2 x + \phi_2, \quad (5.2.14)$$

with $\phi_n \ll \Phi_n$, $n = 1, 2$ and linearize all the conditions. We then get

$$\nabla^2 \phi_1 = 0, \quad z > 0, \quad (5.2.15)$$

and

$$\nabla^2 \phi_2 = 0, \quad z < 0, \quad (5.2.16)$$

Note that

$$f(x, \zeta, t) = f(x, 0, t) + \zeta \frac{\partial f}{\partial z} \Big|_{z=0} + \dots, \quad (5.2.17)$$

On the interface we have the kinematic conditions

$$\frac{\partial \zeta}{\partial t} + U_1 \frac{\partial \zeta}{\partial x} = \frac{\partial \phi_1}{\partial z}, \quad z = 0, \quad (5.2.18)$$

$$\frac{\partial \zeta}{\partial t} + U_2 \frac{\partial \zeta}{\partial x} = \frac{\partial \phi_2}{\partial z}, \quad z = 0, \quad (5.2.19)$$

and the dynamic condition

$$\rho_1 \left(\frac{\partial \phi_1}{\partial t} + U_1 \frac{\partial \phi_1}{\partial x} \right) = \rho_2 \left(\frac{\partial \phi_2}{\partial t} + U_2 \frac{\partial \phi_2}{\partial x} \right), \quad z = 0. \quad (5.2.20)$$

where (5.2.13) has been used.

5.2.3 Normal mode analysis

Assume a sinusoidal disturbance

$$\zeta = Ae^{i(kx-\omega t)} \quad (5.2.21)$$

We follow the usual convention that the real part of the final solution is to be taken to represent physical quantities. To satisfy Laplace equation we take

$$\phi_1 = \bar{\phi}_1 e^{-kz} e^{i(kx-\omega t)} \quad (5.2.22)$$

$$\phi_2 = \bar{\phi}_2 e^{kz} e^{i(kx-\omega t)} \quad (5.2.23)$$

Here $A, \bar{\phi}_1, \bar{\phi}_2$ are unknown constants.

Substituting into boundary conditions on the interface eqs (5.2.18-5.2.20) we get three homogeneous algebraic equations for $A, \bar{\phi}_1, \bar{\phi}_2$,

$$(-i\omega + ikU_1)A = -k\bar{\phi}_1 \quad (5.2.24)$$

$$(-i\omega + ikU_2)A = k\bar{\phi}_2 \quad (5.2.25)$$

$$\rho_1(-i\omega + ikU_1)\phi_1 = \rho_2(-i\omega + ikU_2)\phi_2$$

which can be written in matrix form

$$\begin{bmatrix} -i\omega + ikU_1 & k & 0 \\ -i\omega + ikU_2 & 0 & -k \\ 0 & \rho_1(-i\omega + ikU_1) & -\rho_2(-i\omega + ikU_2) \end{bmatrix} \begin{Bmatrix} A \\ \bar{\phi}_1 \\ \bar{\phi}_2 \end{Bmatrix} = 0$$

In order to get non trivial solutions the coefficient determinant of the algebraic equations must vanish, leading to the eigenvalue condition

$$\rho_1(-\omega + kU_1)^2 + \rho_2(-\omega + kU_2)^2 = 0$$

or

$$(\rho_1 + \rho_2)\omega^2 - 2k(\rho_1U_1 + \rho_2U_2)\omega + k^2(\rho_1U_1^2 + \rho_2U_2^2) = 0$$

The solution is

$$\omega = k \frac{\rho_1U_1 + \rho_2U_2}{\rho_1 + \rho_2} \pm ik \frac{\sqrt{\rho_1\rho_2}|U_1 - U_2|}{\rho_1 + \rho_2} \quad (5.2.26)$$

The lower sign corresponds to instability. Thus a velocity discontinuity is always unstable. This is the limiting case of the continuous velocity profile with a point of inflection.

5.2.4 Physical interpretation

Let us use the mathematical results to see the physics (Batchelor, p 516ff). For simplicity we choose $\rho_1 = \rho_2 = \rho$ and $U_1 = -U_2 = U > 0$ so that (5.2.26) reduces to

$$\omega = \pm ikU \quad (5.2.27)$$

and is purely imaginary. The positive root (upper sign) signifies instability.

The basic flow has a vortex sheet at $z = 0$. Due to the disturbance, the vorticity distribution is changed. Across the interface the total vorticity perturbation is approximately

$$\int_{0_-}^{0_+} \frac{\partial u'}{\partial z} dz = u'_1(z = 0_+) - u'_2(z = 0_-) = \frac{\partial \phi_1}{\partial x} \Big|_0 - \frac{\partial \phi_2}{\partial x} \Big|_0$$

Note that a positive vorticity vector points into the paper (as does the y axis); the circulation is clockwise in the $x - z$ plane. From the solution (5.2.24) and (5.2.25) we get

$$\frac{\partial \phi_1}{\partial x} \Big|_0 = ik\bar{\phi}_1 e^{ikx - i\omega t} = -i(-i\omega + ikU_1)Ae^{ikx - i\omega t} = -i(-i\omega + ikU_1)\zeta$$

and

$$\frac{\partial \phi_2}{\partial x} \Big|_0 = ik\bar{\phi}_2 e^{ikx - i\omega t} = i(-i\omega + ikU_2)Ae^{ikx - i\omega t} = i(-i\omega + ikU_2)\zeta$$

Therefore the vorticity perturbation along the interface is

$$[-2\omega + k(U_1 + U_2)]\zeta = -2\omega\zeta = \mp 2ikU\zeta = 2kUA \exp(ikx \mp i\pi/2) \pm kUt$$

Consider only the upper signs for the unstable solution,

$$-2ikU\zeta = 2kUA \exp \left[ik \left(x - \frac{\pi}{2k} \right) + kUt \right] = 2kUA \exp \left[ik \left(x - \frac{\lambda}{4} \right) + kUt \right]$$

since $k = 2\pi/\lambda$ where λ is the wave length. In real form the vorticity disturbance varies as

$$2kUA \cos \left[k \left(x - \frac{\lambda}{4} \right) \right] e^{kUt}$$

whereas the interface varies as

$$A \cos kx e^{kUt}$$

The vorticity disturbance varies along the interface and leads the interface displacement ζ in phase by $\pi/2$ or one quarter of a wavelength. As sketched in the upper part of Figure 5.2.1, the vorticity disturbance is at the positive (clockwise) maximum (the total vorticity is increased the most) at points like A where $\zeta = 0$ and $\zeta_x < 0$ and at the negative (counterclockwise) maximum (the total vorticity is reduced the most) at points like B where $\zeta = 0$ and $\zeta_x > 0$, being zero at the crests and troughs of the interface. All these vortices tend to lift the interface crests and suppress the troughs.

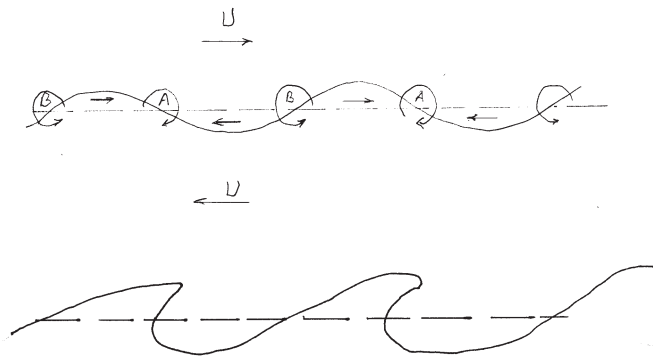


Figure 5.2.1: Disturbance vorticity along a velocity discontinuity. $U_1 = U > 0, U_2 = -U < 0$.

Moreover the disturbances are convected by the average velocity

$$\frac{\partial \phi_1}{\partial x} \Big|_0 + \frac{\partial \phi_2}{\partial x} \Big|_0 = k(U_1 - U_2)\zeta = 2U\zeta$$

which is positive near the crests and negative near the troughs. Therefore clockwise vorticity disturbances accumulate around A , while the counterclockwise disturbances are swept away from and thinned out around B . Because of this asymmetry the wavy interface not only tends to amplify but to curl up in the forward direction as sketched in the lower part of Figure 5.2.1.

After some time the disturbance grows so large that nonlinearity must be accounted for. Numerical nonlinear analysis as well as laboratory experiments show that the interface curls up, as can be seen in the famous photograph by Thorpe: Figure 5.2.2.

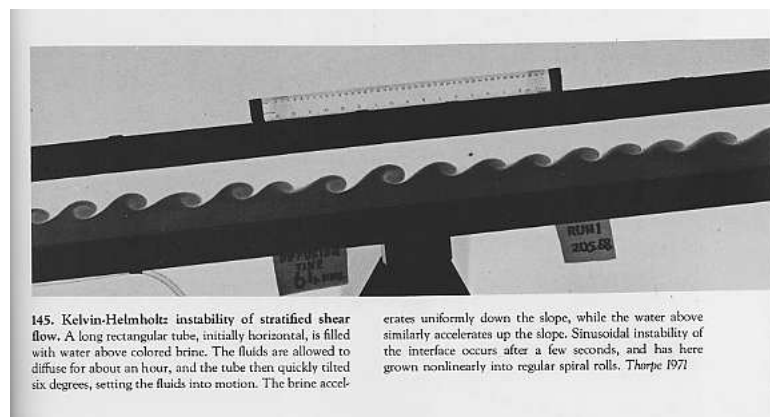


Figure 5.2.2: Kelvin-Helmholtz instability in the laboratory.

Homework K-H instability: Show that when gravity and surface tension are both present, the eigenvalue condition is

$$\frac{\omega}{k} = \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} \pm \left[-\frac{\rho_1 \rho_2 (U_1 - U_2)^2}{(\rho_1 + \rho_2)^2} + \frac{g \rho_2 - \rho_1}{k \rho_1 + \rho_2} + \frac{Tk}{\rho_1 + \rho_2} \right]^{1/2} \quad (5.2.28)$$

Discuss the condition for instability.

Remark: The special case of $U_1 = U_2 = 0$ but $g \neq 0$, i.e., the instability of normally accelerated density discontinuity, is called the Rayleigh-Taylor instability. It is of interest when an interface is accelerated in the normal direction, and has a wide range of industrial and astrophysical applications.

Remark: The K-H instability problem can be generalized in a number of ways, by including viscosity and continuous density and/or velocity profiles, see Chandrasekhar.

Summary: Let us summarize the standard procedure of linearized instability analysis as follows:

1. Assume the flow domain to be of infinite extent in the x direction. Let the basic flow have velocity $U(z)$ in the x direction.
2. Derive the linearized equation for infinitesimal perturbations and homogeneous boundary conditions.
3. Consider sinusoidal wave-like disturbances so that all unknowns are of the form

$$f_i = \bar{f}_i(y, z) e^{i(kx - \omega t)}$$

4. Deduce the partial differential equation(s) in the transverse plane (y, z) and obtain an eigenvalue problem.
5. Solve for the eigenvalue ω for given real k .
6. If the eigenvalue is complex, $\omega = \omega_r + i\omega_i$, find the condition under which $\omega_i > 0$. Since

$$e^{-i\omega t} = e^{-i(\omega_r + i\omega_i)t} = e^{-i\omega_r t} e^{\omega_i t},$$

the flow is unstable $\omega_i > 0$ as time grows, stable if $\omega_i < 0$, and neutrally stable if $\omega_i = 0$.