Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
by Chiang C. Mei, 2007

5.3 Inviscid instability mechanism of parallel flows

We now turn to an older problem of the instability of parallel flow without stratification and gravity, such as channel flows, jets, wakes and boundary layers.

5.3.1 Rayleigh's equation

In a shear flow a necessary conditon for instablity is that there must be a point of inflection in the velocity profile $U(z)$, i.e,

$$\frac{d^2U}{dz^2} = 0 \tag{5.3.1}$$

somewhere in the flow.

Let us begin with an inviscid uniform flow.

$$\vec{U} = U\vec{i}, \quad P = 0 \tag{5.3.2}$$

and add an infinitesimal disturbance so that the total velocity and pressure fields are

$$(u, w, p) = (U + u', w', p') \tag{5.3.3}$$

where primes denote disturbances which are functions of (x, z, t). The governing equations are

$$u'_x + w'_z = 0 \tag{5.3.4}$$

$$\rho(u'_t + (U + u')u'_x + w'(U'_z + u'_z) = -p'_x \tag{5.3.5}$$

$$\rho(w'_t + (U + u')w'_x + w'w_z = -p'_z \tag{5.3.6}$$

The bounddray conditions are

$$w' = 0, \quad z = 0, d \tag{5.3.7}$$

where d can be unbounded.

Linearizing the momentum equations by omitting terms quadratic in disturbances, we get

$$\rho\left(u'_t + Uu'_x + \frac{dU}{dz}w'\right) = -p'_x \tag{5.3.8}$$

$$\rho(w'_t + Uw'_x) = -p'_z \tag{5.3.9}$$

Eliminating p' by cross-differntiation,

$$\rho\left(u'_t + Uu'_x + \frac{dU}{dz}w'\right)_z = -p'_{xz} = \rho(w'_t + Uw'_x)_x$$

Introducing the stream function by $u' = \psi_z, w' = -\psi_x$, we get

$$\left(\psi_{zt} + U\psi_{xz} - \frac{dU}{dz}\psi_x\right)_z = (-\psi_{xt} + U\psi_{xx})_x$$

Let us consider a wave like disturbance

$$\psi = f(z)e^{ik(x-Ct)}, \quad \text{where } \; C = \frac{\omega}{k} \tag{5.3.10}$$

then

$$\left(-ikCf_z + ikUf_z - ik\frac{dU}{dz}f\right)_z = ik\left(k^2Cf + k^2Uf\right)$$

or

$$(U(z) - C)\left(f_{zz} - k^2 f\right) - U_{zz}f = 0. \tag{5.3.11}$$

which is known as Rayleigh's equation in hydrodynamic instability. The boundary conditions are:

$$f = 0, \quad z = 0, d \tag{5.3.12}$$

5.3.2 Rayleigh's necessary condition for instability

Let us rewrite (5.3.11) as

$$f_{zz} - k^2 f - \frac{U_{zz}}{U - C}f = 0 \tag{5.3.13}$$

and multiply by the complex conjugate

$$f^*\left(f_{zz}\right) - k^2 ff^* - \frac{U_{zz}}{U - C}ff^* = 0$$

which can be rewritten by partial integration

$$\frac{d}{dz}(f^* f_z) - |f_z|^2 - k^2|f|^2 - \frac{U_{zz}}{U - C}|f|^2 = 0$$

Integrating from $z = 0$ to $z = d$ and applying the boundary conditions we get simply

$$\int_0^d dz \frac{U_{zz}}{U - C}|f|^2 = -\int_0^d dz\left(|f_z|^2 + k^2|f|^2\right) \tag{5.3.14}$$

Now we let $C = C_r + iC_i$ to be complex, then on the left-hand side,

$$\frac{U_{zz}|f|^2}{U - C} = \frac{U_{zz}(U - C^*)|f|^2}{|U - C|^2} = \frac{U_{zz}(U - C_r + iC_i)|f|^2}{|U - C|^2}$$

Equation (5.3.14) becomes

$$\int_0^d dz \frac{U_{zz}(U - C_r + iC_i)|f|^2}{|U - C|^2} = -\int_0^d dz\left(|f_z|^2 + k^2|f|^2\right) \tag{5.3.15}$$

We get from the imaginary part of (5.3.15),

$$\int_0^d dz \frac{U_{zz}|f|^2}{|U-C|^2} = 0 \tag{5.3.16}$$

if $C_i \neq 0$. Thus for instability it is necessary that U_{zz} is positve for some y and negative elsewhere, i.e., U_{zz} must vanish somewhere in the flow, i.e., the velocity profile must have an inflection point inside the flow. This is Rayleigh's necessary condition for instability.

5.3.3 Physical explanation (Lin, 1945)

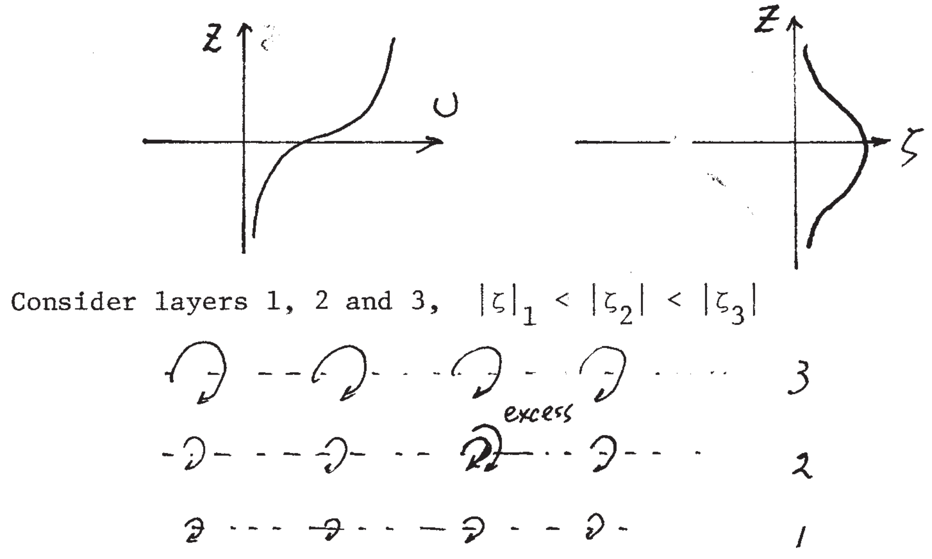

Figure 5.3.1: (a) Velocity profile with an inflection point. (b) Vorticity profile. (c) Effect of exchanging fluid parcels on vorticity

In Figure 5.3.1 we sketch the velocity profile with a inflection point at $z = 0$ and the corresponding profile of vorticity $\zeta = U_z$ which has a maximum at $z = 0$. (Note the in the right-handed system the y axis points into the paper, therefore the flow of a positive vortex is clockwise.

Consider three layers in the region where $U(z)$ increases with z monotonically, i.e., below the inflection point, with $\zeta(z_1) < \zeta(z_2) < \zeta(z_3)$. If a fluid parcel descends from level 3 to level 2, it brings with its vorticity without change, according the vorticity transport law in an inviscid fluid. An excess vorticity is created at level 2 which tends to replace the fluid on the right in layer 2 by fluid with higher vorticity, and the fluid on the left in layer 2 by fluid with lower voriticity. The net consequence is to force the original excess vortex to return to layer 3. Similarly if a fluid parcel ascends from level 1 to level 2, it brings with its vorticity, hence creates an excess vorticity defect with tends to replace the fluid on the left in layer 2 by fluid with higher vorticity, and the fluid on the right in layer 2 by fluid with lower vorticity. The net consequence is to force the original defect votex to return to layer 3. Thus an accidental diplacement of a fluid parcel tends to it original level; the flow is stable.

By a similar reasoning, if $U(z)$ decreases monotonically in z the flow is also stable. However, if there is a level of vorticity extremum, say level 0 at $z = 0$, then a fluid element arriving at this layer is not forced back to its origin. A fluid element on one side of level 0 is equally at home on the opposite side. The flow is unstable.

5.3.4 Fjortoft's stronger condition

Further information can be obtained from the real part of (5.3.15),

$$\int_0^d dz \left(\frac{U_{zz}(U - C_r)}{|U - C|^2}\right)|f|^2 = -\int_0^d dz \left(|f_z|^2 + k^2|f|^2\right) \tag{5.3.17}$$

Now let U_I be the velocity at the point of inflection, we use (5.3.16) so that

$$\int_0^d dz \frac{U_{zz}(U_I - C_r)|f|^2}{|U - C|^2} = (U_I - C_r)\int_0^d dz \frac{U_{zz}|f|^2}{|U - C|^2} = 0 \tag{5.3.18}$$

The difference of (5.3.17) and (5.3.18) is

$$\int_0^d dz \frac{U_{zz}(U - U_I)}{|U - C|^2}|f|^2 = -\int_0^d dz \left(|f_z|^2 + k^2|f|^2\right) < 0 \tag{5.3.19}$$

If $U(z)$ is a monotonic function with one inflection point, a necessary condition for instability is that the product

$$U_{zz}(U - U_I) < 0 \tag{5.3.20}$$

for all z in the flow.

Referring to Figure 5.3.2, there is no point of inflection in flows in (a) and (b) hence do not satisfy Raleigh's necessary criterion for instability. The flow in (c) does not satisfy Fjortoft critierion since $U_{zz}(U - U_I) > 0$. Only (d) which is representative of jets and wakes, satisfies both Rayleigh anbd Fjortoft theorems.

Poiseuille flows in a pipe and boundary layers on a flat plate do not satisfy the Rayleigh-Fjortoft criterion. Why then is Poiseuille flow known to be unstable beyond $Re = 2100$ (Reynolds)?

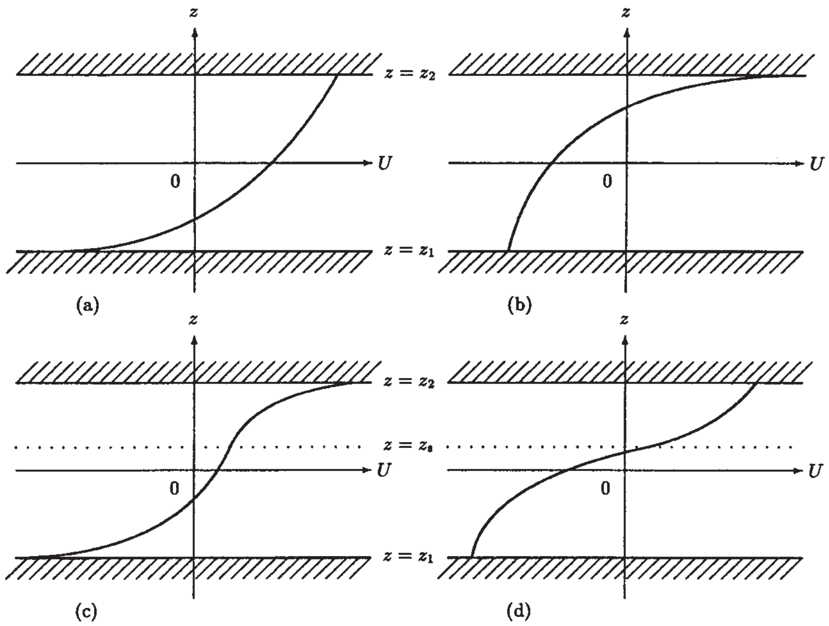

Figure 8.3 Some examples of flows governed by the Rayleigh–Fjørtoft necessary conditions for instability. (a) Stable because $U'' < 0$ everywhere. (b) Stable because $U'' > 0$ everywhere. (c) Stable because $U''(z_s) = 0$ but $U''(U - U_s) \geq 0$. (d) Possibly unstable because $U''(z_s) = 0$ and $U''(U - U_s) \leq 0$.

Figure 5.3.2: Fjortoft Theorm , From Drazin 2002.