Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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CHAPTER 6.
SEEPAGE AND THERMAL EFFECTS IN POROUS MEDIA

6-1darcy-EM.tex April 16, 2004

Applications : Groundwater flow and transport, building insulation, energy storage and recovery, geothermal reservoirs, nuclear waste disposal,etc.

6.1 Empirical basis of Darcy's law for seepage flow

[References]:

Polubarinova-Kochina: *Theory of Groundwater Movement*, Princeton University Press

The basis of Darcy's law for a non-deformable medium is the one dimensional experiment by Darcy, see Figure 6.1.1.

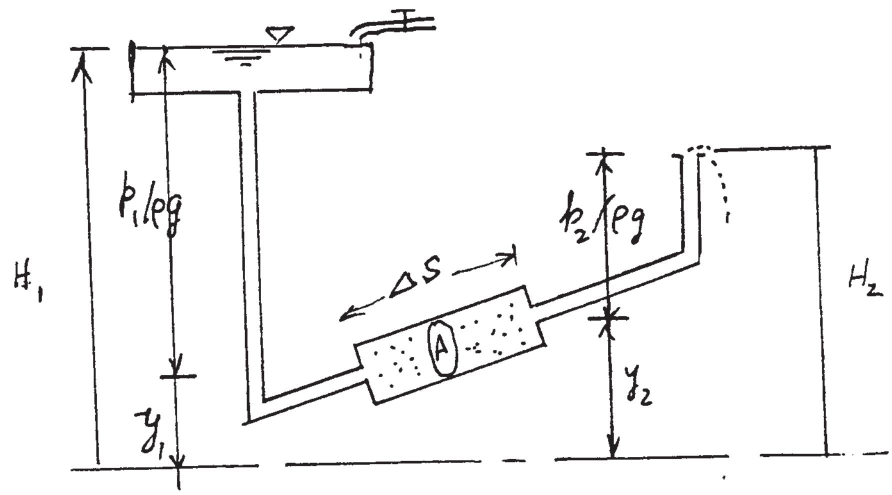

Figure 6.1.1: Darcy's experiment for seepage flow

The discharge through the tube is measured to be

$$Q = KA\frac{H_1 - H_2}{\Delta s} \tag{6.1.1}$$

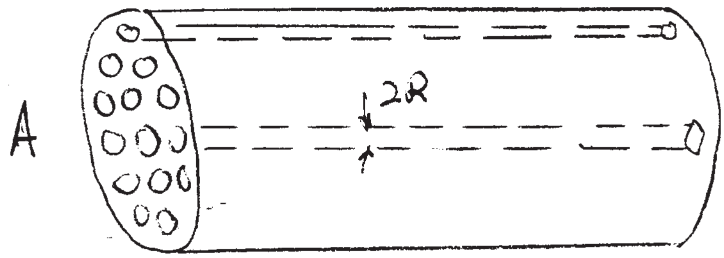

Figure 6.1.2: A one dimensional model of porous medium

where

$$H_1 = \frac{P_1}{\rho g} + y_1 \qquad H_2 = \frac{P_2}{\rho g} + y_2$$

and K is an empircal coefficient called the hydraulic conductivity. The effective (seepage, or filtration) velocity is defined to be the discharge per unit gross area of the porous medium

$$\bar{u} = \frac{Q}{A} = -K\frac{H_2 - H_1}{\Delta s} \tag{6.1.2}$$

Let us define the potential to be

$$\phi = -K\left(\frac{p}{\rho g} + y\right) = -KH \tag{6.1.3}$$

then

$$\phi_i = -K\left(\frac{p_i}{\rho g} + y_i\right) = -KH_i, \quad i = 1, 2, \tag{6.1.4}$$

In the limit of $\Delta s \to 0$, we get

$$\bar{u} = -K\frac{\partial H}{\partial s} = \frac{\partial \phi}{\partial s} \tag{6.1.5}$$

This is Darcy's empirical law relating the seepage velocity to the hydraulic heads, both are macro-scale averaged quantities.

What affect the conductivity? Let the porosity n be defined as the percentage of pore volume in the gross volume V. If V_s is the volume occupied by solid grains in V, then

$$n = \frac{V - V_s}{V} \tag{6.1.6}$$

If the pores are satuarated with fluid then $V_f = V - V_s$ and $n = V_f/V$.

Consider the idealized porous medium consisting of parallel tubes, Figure 6.1.2. In a cross-section of area A, the net area of pores is nA. The net averaged velocity is

$$\bar{u}_P = \frac{Q}{nA} = \frac{\bar{u}}{n} \tag{6.1.7}$$

For a laminar flow through a circular tube, the discharge is

$$Q = -\frac{\pi R^4}{8\mu\Delta s}(\Delta p + \Delta y \cdot \rho g)$$

(Homework). The averaged velocity in the pore (tube) is

$$\bar{u}_P = \frac{Q}{\pi R^2} = -\frac{R^2 g}{8\nu}\frac{(\Delta p/\rho g + \Delta y)}{\Delta s}$$

The seepage velocity through the matrix is

$$n\,\bar{u}_P = -\frac{nR^2 g}{8\nu}\frac{(\Delta p/\rho g + \Delta y)}{\Delta s} \tag{6.1.8}$$

Therefore, the hydraulic conductivity is

$$K = \frac{nR^2 g}{8\nu} \tag{6.1.9}$$

with the dimension

$$[K] = \frac{L}{T} \tag{6.1.10}$$

The real pores are, of course, geometrically more complex, but the preceding formula indicates that K is small for small pores and for high viscosity.

Darcy's law is also often expressed in the form,

$$\bar{u} = -\frac{k}{\mu}(p + \rho g y) \tag{6.1.11}$$

where k is called the (intrinsic) permeability. Clearly

$$k = \frac{\mu K}{\rho g} \tag{6.1.12}$$

For the tubular model we have

$$k = \frac{nR^2}{8} \tag{6.1.13}$$

which is independent of viscosity.

From experiments, Kozeny and Carman proposed the following (empirical) formula

$$k = cd^2\frac{n^3}{(1-n)^2} \tag{6.1.14}$$

where $c = 0.1 \sim 0.8$, and d is the effective pore diameter defined as the ratio of the volume of solids to the wetted area in the gross volume.

Materials	Hydraulic Conductivity K(m/sec)
Clays	$< 10^{-9}$
Sandy clays	$10^{-9} - 10^{-8}$
Peat	$10^{-9} - 10^{-7}$
Silt	$10^{-8} - 10^{-7}$
Very fine sands	$10^{-6} - 10^{-5}$
Fine sands	$10^{-5} - 10^{-4}$
Coarse sands	$10^{-4} - 10^{-3}$
Sand with gravel	$10^{-3} - 10^{-2}$
Gravels	$> 10^{-2}$

Table 6.1: The order of magnitude of the conductivity of natural soils

Three-dimensional Darcy law; As a generalization

$$\bar{u}_i = -K_{ij}\frac{\partial H}{\partial x_j} = -\frac{k_{ij}}{\mu}\frac{\partial(p+\rho g y)}{\partial x_j} \tag{6.1.15}$$

where K_{ij} denotes the conductivity tensor and k_{ij} the permeability tensor. For an macroscopically isotropic material

$$K_{ij} = K\delta_{ij}, \quad k_{ij} = k\delta_{ij} \tag{6.1.16}$$

Continuity requires that

$$\nabla \cdot \vec{\bar{u}} = 0 \tag{6.1.17}$$

Thus

$$\frac{\partial}{\partial x_i}\left(K_{ij}\frac{\partial H}{\partial x_j}\right) = 0 \tag{6.1.18}$$

in general, and

$$\nabla \cdot K\nabla\phi = 0 \tag{6.1.19}$$

for isotropic media. If further, the material is homogeneous : K = constant. then

$$\nabla^2\phi = 0 \tag{6.1.20}$$

Hence, for a nondeformable isotropic and homogeneous porous medium, the flow is potential.

In most soils the pore flow is usually laminar. Take the typical values : $u = 0.25cm/sec, d \sim 0.4\ mm$, then

$$Re = \frac{uD}{\nu} = \frac{0.01cm^2/sec}{0.01\ cm^2/sec} = 0.1.$$

It is known empirically that for $Re < 1 \sim 15$, the flow is usually laminar.

Boundary conditions of a typical seepage problem. Consder an earth dam:

On the soil water interface $y = H_1(x)$ AB:

$$p = p_a + \rho g \left(H_1 - y\right)$$

$$\phi = -K\left(\frac{p}{\rho g} + y\right) = -K\left(\frac{p_a}{\rho g} + H_1 - y + y\right) = -K\left(\frac{p_a}{\rho g} + H_1\right) = \text{constant}$$

Therefore,

$$\phi = \text{constant}, \quad y = H_1(x). \tag{6.1.21}$$

On the phreatic surface AE, where $y = Y(x)$ is unknown a priori. The dynamic condition is $p =$ constant. Therefore,

$$\phi + KY(x) = \text{constant}, \quad y = Y(x) \tag{6.1.22}$$

In addition, we have the kinematic condition, that $y = Y(x)$ is a streamline :

$$\psi = \text{constant}, \quad y = Y(x) \tag{6.1.23}$$

On the seepage surface ED:

$$p = \phi + Ky = \text{constant} = p_a, \quad y = H_2(x) \tag{6.1.24}$$

where $H_2(x)$ is known. Note that just inside the soil, the fluid velocity is not tangential to ED.

On the impervious boundary (rock or saturated fine clay, etc.), BF:

$$\psi = \text{constant} \tag{6.1.25}$$

Because the phreatic surface is unknown, the boundary-value problem in the earth dam is highly nonlinear and difficult.

The prediction of the phreatic surface is useful for the design of the dam thickness and of the size and location of the drainage ditch. To find the stresses in the dam and on the foundation, one should consider the deformation of soil; this requires soil mechanics.

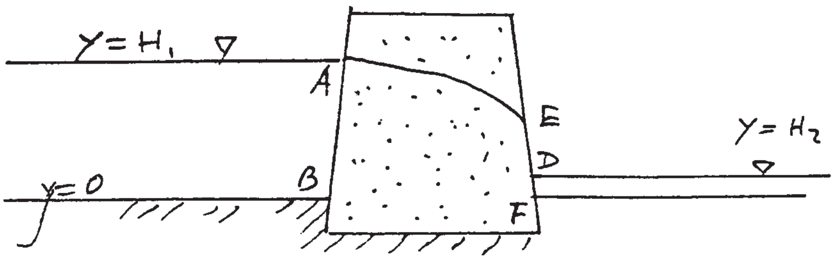

Figure 6.1.3: A one dimensional model of porous medium