**Lecture Notes on Fluid Dynamics**
(1.63J/2.21J)
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chapter

## 7.2 Taylor -Proudman theorem and Vorticity in inviscid rotating fluids

We first show that in a steady rotating flow of inviscidand homogeneous fluid, if the Rossby number is smal, then the flow is essentially two diemensional. This is known as the Taylor-Proudman theorm.

Under these conditions, the momentum equation reads,

$$2\vec{\Omega}\times\vec{q} = -\frac{\nabla p}{\rho} \tag{7.2.1}$$

Taking the curl of both sides we get

$$\nabla\times(\vec{\Omega}\times\vec{q}) = 0 \tag{7.2.2}$$

Using the identity

$$\nabla\times\left(\vec{A}\times\vec{B}\right) = \vec{A}\nabla\cdot\vec{B} - \vec{B}\nabla\cdot\vec{A} + \vec{B}\cdot\nabla\vec{A} - \vec{A}\cdot\nabla\vec{B} \tag{7.2.3}$$

we get

$$\vec{\Omega}\nabla\cdot\vec{q} - \vec{q}\nabla\cdot\vec{\Omega} + \vec{q}\cdot\nabla\vec{\Omega} - \vec{\Omega}\cdot\nabla\vec{q} = 0$$

Invoking continuity and the constancy of $\Omega$ we obtain

$$\vec{\Omega}\cdot\nabla\vec{q} = 0 \tag{7.2.4}$$

Thus the velocity field does not vary in the direction of $\Omega$, say $z$. Note that $\vec{q}$ can still have three components, but they must all be independent of $z$. This is the

**Theorem 1** *Taylor-Proudman theorem : A steady and slow flow in a rotating fluid is two-dimensional in the plane perpendicular to the vector of angular velocity.*

Laboratory verification has been demonstrated in a setup shown in figure 7.2.1.

More generally, let us consider the vortity transport in a rotating and invoscid fluid. Let $\vec{\zeta} = \nabla\times\vec{q}$ and use the identity

$$\vec{\zeta}\times\vec{q} = \vec{q}\cdot\nabla\vec{q} - \nabla\frac{|\vec{q}|^2}{2}$$

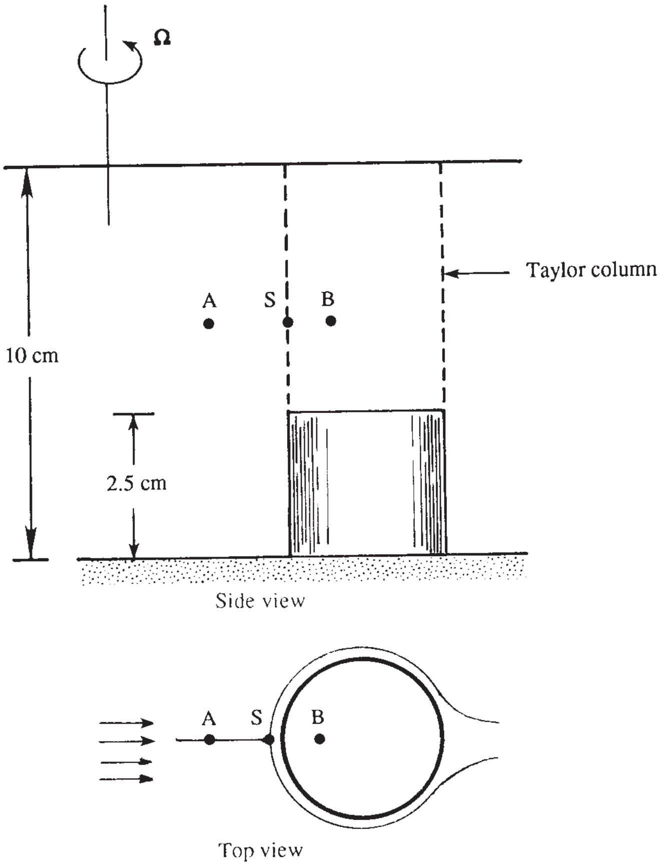


**Fig. 13.6** G. I. Taylor's experiment in a strongly rotating flow of a homogeneous fluid.

Figure 7.2.1: Taylor's experiment showing the Taylor column above a truncated cylinder in a rotating fluid. The large container with water rotates but the cylinder is fixed in space. From Kundu.

The mommentum equation can be written :

$$\frac{\partial \vec{q}}{\partial t} + \vec{\zeta} \times \vec{q} + 2\vec{\Omega} \times \vec{q} = -\frac{\nabla p}{\rho} + \nabla \left( \phi - \frac{|\vec{q}|^2}{2} \right) \tag{7.2.5}$$

Taking the curl of the above equation:

$$\frac{\partial \vec{\zeta}}{\partial t} + \nabla \times \left( (2\vec{\Omega} + \vec{\zeta}) \times \vec{q} \right) = \frac{\nabla \rho \times \nabla p}{\rho^2}$$

Using the identity (7.2.3), we get

$$\nabla \times \left\{ (2\vec{\Omega} + \vec{\zeta}) \times \vec{q} \right\} = -\vec{q} \nabla \cdot (2\vec{\Omega} + \vec{\zeta}) + (2\vec{\Omega} + \vec{\zeta}) \nabla \cdot \vec{q} + \vec{q} \cdot \nabla (2\vec{\Omega} + \vec{\zeta}) - (2\vec{\Omega} + \vec{\zeta}) \cdot \nabla \vec{q}$$

The first term on the right vanishes because $\vec{\Omega}$ = constant and the divergence of curl is zero; the second vanishes for imcompressible fluids. Let $\vec{\zeta}_a = \vec{\zeta} + 2\vec{\Omega}$ = absolute vorticity

$$\frac{D\vec{\zeta}}{Dt} = \frac{\partial \vec{\zeta}}{\partial t} + \vec{q} \cdot \nabla \vec{\zeta} = \vec{\zeta}_a \cdot \nabla \vec{q} + \frac{\nabla \rho \times \nabla p}{\rho^2} \tag{7.2.6}$$

In a fluid of constant density and a steady flow of small Rossby number

$$\epsilon = \text{Rossby No.} = \frac{u}{2\Omega L} \ll 1$$

then

$$\frac{\zeta}{2\Omega} =\sim \frac{u}{2\Omega L} \ll 1$$

(7.2.6) reduces to (7.2.4).