Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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7.3 The Shallow-Water Approximation

For simplicity we shall demonstrate the reasoning only for a shallow layer of inviscid fluid of constant depth.

The horizontal length scale of motion L is assumed to be much greater than the sea depth D,

$$\frac{D}{L} \ll 1 \tag{7.3.1}$$

by contiunity, The continuity equation reads

$$\underbrace{\frac{\partial u}{\partial x}}_{\frac{U}{L}} + \underbrace{\frac{\partial v}{\partial y}}_{\frac{U}{L}} + \underbrace{\frac{\partial w}{\partial z}}_{\frac{W}{D}} = 0$$

From this we can infer first that

$$W = \frac{DU}{L} \ll U \tag{7.3.2}$$

Let us recall the approximation,

$$2\vec{\Omega} \times \vec{q} \approx -fv\vec{i} + fu\vec{j} \tag{7.3.3}$$

and ignore friction so that the momentum equations are

$$\underbrace{\frac{\partial u}{\partial t}}_{\frac{U}{T}} + (\underbrace{u\frac{\partial u}{\partial x}}_{\frac{U^2}{L}} + \underbrace{v\frac{\partial u}{\partial y}}_{\frac{U^2}{L}} + \underbrace{w\frac{\partial u}{\partial z}}_{\frac{U^2}{L}}) - \underbrace{fv}_{fU} = \underbrace{\frac{1}{\rho}\frac{\partial p_d}{\partial x}}_{\frac{P}{\rho L}}$$

$$\frac{\partial v}{\partial t} + \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) + fu = -\frac{1}{\rho}\frac{\partial p_d}{\partial y},$$

$$\underbrace{\frac{\partial w}{\partial t}}_{\frac{W}{T}} + (\underbrace{u\frac{\partial w}{\partial x}}_{\frac{UW}{L}} + \underbrace{v\frac{\partial w}{\partial y}}_{\frac{UW}{L}} + \underbrace{w\frac{\partial w}{\partial z}}_{\frac{UW}{L}}) = \underbrace{\frac{1}{\rho}\frac{\partial p_d}{\partial z}}_{\frac{P}{\rho D}}$$

where p_d stands for the dynamic pressure

$$p = -\rho g z + p_d$$

The boundary condition on the sea surface is that

$$p = p_{atm}, \quad z = \eta(x, y, t) \tag{7.3.4}$$

and that the normal velocity of the fluid equals the normal volocity of the surface,

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = w, \quad z = \eta \tag{7.3.5}$$

The buundary condition on the bottom is

$$w = 0 \tag{7.3.6}$$

Let us focus attention to situations where Coriolis acceleration and instantaneous acceleration are comparably important. In order that the pressure gradient can drive the flow, we require the dynamic pressure scale to be

$$P_d = \frac{\rho U L}{T}, \quad \text{or} \quad \rho U f L$$

From the vertical momentum equation we estimate

$$\frac{\frac{\partial w}{\partial t}}{\frac{1}{\rho}\frac{\partial p_d}{\partial z}} \sim \frac{\frac{DU}{LT}}{\frac{\rho U L/T}{\rho D}, \frac{\rho U f L}{\rho D}} \sim \frac{D^2}{L^2}\left(1, \frac{1}{fT}\right)$$

For storm surges, the time scale of interest is of a day or so, $fT = O(1)$. We conclude that the vertical pressure gradient is dominates the vertical momentum balance with an error of order $D^2/L^2 \ll 1$.

Equating the total pressure on the free surface at $z = \zeta$ to the atmospheric pressure,

$$p_{\text{total}} \simeq p_{atm}(x, y, t) + \rho g(\eta - z), \quad p_d \simeq p_{atm} + \rho g \eta. \tag{7.3.7}$$

i.e., the total pressure is hydrostatic In particular if the atmospheric pressure is constant, we have simply

$$\frac{\partial p_d}{\partial x} = \rho g \frac{\partial \eta}{\partial x} \qquad \frac{\partial p_d}{\partial y} = \rho g \frac{\partial \eta}{\partial y} \tag{7.3.8}$$

Thus, we have

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{7.3.9}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - fv = -g\frac{\partial \eta}{\partial x} \tag{7.3.10}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + fu = -g\frac{\partial \eta}{\partial y} \tag{7.3.11}$$

By integrating the continuity condition across the entire depth and making use of the kinematic boundary condtions we get the depth-integrated mass conservation law,

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\int_{-h}^{\eta} u\,dz + \frac{\partial}{\partial y}\int_{-h}^{\eta} v\,dz = 0 \tag{7.3.12}$$

We shall show later that real fluid effects are limited in the Ekman boundary layer which is often small compared to the sea depth. It is natural to expect that the horizontal velocity u, v to depend weakly on z i.e., uniform in depth, It follows that

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[(\eta + h)u] + \frac{\partial}{\partial y}[(\eta + h)v] = 0 \tag{7.3.13}$$

and

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial x} \tag{7.3.14}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y} \tag{7.3.15}$$

If Rossby number is small, then convective inertia is also negligible; the momentum equations reduce to :

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{7.3.16}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{7.3.17}$$

where are linear.

Take for estimates, $U = 0.1, 1m/s, \Omega = 2.31 \times 10^{-5}, L = 100, 1000km = 10^5, 10^6 m$, Rossby number $U/2\Omega L = 0.043$. We leave it as an exercise to work out the equations for $h(x, y)$ with a small slope.

7.3.1 Geostrophic motion

For steady flow at small Rossby number,

$$H = \eta + h \simeq h$$

the momentum equations reduce to

$$\begin{aligned} -fv &= -g\frac{\partial \eta}{\partial x} \\ fu &= -g\frac{\partial \eta}{\partial y}. \end{aligned}$$

Thus, Coriolis force and pressure gradient are in balance

$$u = -\frac{g}{f}\frac{\partial \eta}{\partial y} \qquad v = \frac{g}{f}\frac{\partial \eta}{\partial x}$$

implying

$$u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = 0 \tag{7.3.18}$$

or

$$\vec{q} \cdot \nabla \eta = 0.$$

Physically along a streamlines, the free surface height remains constant. Hence, the surface contours are parallel to the streamlines and to isobars. This state is called **geostrophic**.

7.3.2 Appendix 1: Depth-integrated mass conservation

The depth integrated mass conservation (7.3.29) is a general an exact result which holds for variable depth as well. On the seabed $z = -h(x, y)$, vanishing of the normal veocity requires

$$w = -u\frac{\partial h}{\partial x} - v\frac{\partial h}{\partial y}.$$

On the free surface $z = \eta(x, y, t)$, the kinematic boundary condition reads,

$$w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} \qquad z = \eta$$

Integrating the continuity equation and using Leibniz's rule:

$$\frac{\partial}{\partial x}\int_{-b(x)}^{a(x)} f(x, z)dz = \int_{-b}^{a} f\frac{\partial f}{\partial x}dz + \frac{da}{dx}f(x, a(x)) + \frac{db}{dx}f(x, -b(x)) \tag{7.3.19}$$

then

$$\begin{aligned} 0 &= \int_{-h}^{\eta}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dz \\ &= [w]_{-h}^{\eta} + \frac{\partial}{\partial x}\int_{-h}^{\eta} udz + \frac{\partial}{\partial y}\int_{-h}^{\eta} vdz \\ &\quad -\frac{\partial \eta}{\partial x}u(\eta) - \frac{\partial \eta}{\partial y}v(\eta) - \frac{\partial h}{\partial x}u(-h) - \frac{\partial h}{\partial y}v(-h). \end{aligned}$$

or,

$$0 = \left(w - u\frac{\partial \eta}{\partial x} - v\frac{\partial \eta}{\partial y}\right)_{\eta} - \left(w + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial x}\right)_{-h} + \frac{\partial[(\eta + h)\bar{u}]}{\partial x} + \frac{\partial[(\eta + h)\bar{v}]}{\partial y}$$

Using the boundary conditions we get

$$\frac{\partial \eta}{\partial t} + \frac{\partial[(\eta + h)\bar{u}]}{\partial x} + \frac{\partial[(\eta + h)\bar{v}]}{\partial y} = 0. \tag{7.3.20}$$

where

$$(\bar{u}, \bar{v}) = \frac{1}{\eta + h}\int_{-h}^{\eta}(u, v)\, dz$$

denotes the depth-averaged velocity.

7.3.3 Remark 2: Formal perturbation theory

The results (7.3.13), (7.3.14) and (7.3.15) can be confirmed by a formal perturbation scheme:

$$u = u_0 + \frac{z + h}{L}\, u_1 + \frac{(z + h)^2}{2L^2}\, u_2 + \cdots \tag{7.3.21}$$

$$v = v_0 + \frac{z+h}{L} v_1 + \frac{(z+h)^2}{2L^2} v_2 + \cdots \tag{7.3.22}$$

$$w = \frac{z+h}{L} w_1 + \frac{(z+h)^2}{2L^2} w_2 + \cdots \tag{7.3.23}$$

where u_n, v_n, w_n are independent of z. From the x momentum equation we get from (7.3.14)

$$\begin{aligned} &\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} - f v_0 + g \frac{\partial \eta}{\partial x} \\ &\quad + \frac{(z+h)}{L} \left(\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_0}{\partial y} + v_0 \frac{\partial u_1}{\partial y} + \frac{w_1 u_1}{L} - f v_1 \right) \\ &\quad + \left(\frac{z+h}{L} \right)^2 (\cdots) + \cdots = 0 \end{aligned} \tag{7.3.24}$$

with a similar equation for the y momentum. Separating the zeroth power of $(z+h)/L$ we get

$$\frac{\partial u_0}{\partial t} + u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial y} - f v_0 = -g \frac{\partial \eta}{\partial x} \tag{7.3.25}$$

$$\frac{\partial v_0}{\partial t} + u_0 \frac{\partial v_0}{\partial x} + v_0) \frac{\partial v_0}{\partial y} + f u_0 = -g \frac{\partial \eta}{\partial y} \tag{7.3.26}$$

Thus u, v are depth-independent to the leading order. Also from continuity,

$$\frac{\partial u_0}{\partial x} + \frac{\partial v_0}{\partial y} + \frac{w_1}{L} = 0 \tag{7.3.27}$$

and from the free surface condition

$$\frac{\partial \eta}{\partial t} + u_0 \frac{\partial \eta}{\partial x} + v_0 \frac{\partial \eta}{\partial y} = \frac{\eta + h}{L} w_1 \tag{7.3.28}$$

Hence after eliminating w_1,

$$\frac{\partial \eta}{\partial t} + \frac{\partial [(\eta + h) u_0]}{\partial x} + \frac{\partial [(\eta + h) v_0]}{\partial y} = 0 \tag{7.3.29}$$

In summary, for shallow seas,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - f v = -g \frac{\partial \eta}{\partial x} \tag{7.3.30}$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + f u = -g \frac{\partial \eta}{\partial y} \tag{7.3.31}$$

to leading order.