

Lecture Notes on Fluid Dynamics

(1.63J/2.21J)

by Chiang C. Mei, 2004

7-5cyclone.tex

Ref: D. J. Acheson: Elementary Fluid Mechanics, §8.5

7.5 Cyclonic current forced by a swirling wind

We now give an example where there is horizontal nonuniformity. Of practical interest is the case of nonuniform wind stress on the surface. As an extremely simplified model we consider a vortical wind stress over a large sea¹. See Figure 7.5.1.

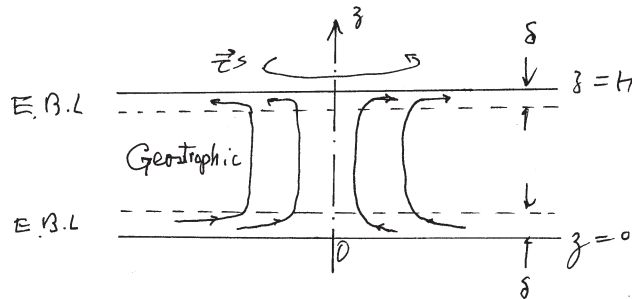


Figure 7.5.1: Steady cyclonic flow in a shallow sea forced by swirling wind

Let us restrict to a low Rossby number flow for simplicity. Continuity requires:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \quad (7.5.1)$$

The momentum equations are

$$-fv = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \nabla^2 u \quad (7.5.2)$$

$$fu = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \nabla^2 v \quad (7.5.3)$$

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \nabla^2 w \quad (7.5.4)$$

¹Acheson demonstrated a very similar problem of a circular layer of water bounded above and below by two horizontal planes. While the bottom plane rotates about the vertical axis at the rate Ω the top cover rotates steadily at a different rate $(1 + \epsilon)\Omega$.

The boundary conditions are : no slip on the bottom:

$$u = v = w = 0, \quad z = 0 \quad (7.5.5)$$

and given wind stress on the top:

$$\tau_{\theta z}^S = \rho T r / 2, \quad \tau_{rz}^S = 0, \quad z = H. \quad (7.5.6)$$

The wind stress is cyclonic, where T is the curl of the wind stress vector (divided by ρ):

$$\nabla \times \vec{\tau}^S = \vec{k} \left(\frac{1}{r} \frac{\partial}{\partial r} (r \tau_{\theta z}^S) - \frac{1}{r} \frac{\partial \tau_{rz}^S}{\partial \theta} \right) = \rho T \vec{k}. \quad (7.5.7)$$

It has the dimension of L/T^2 . In cartesian coordinates the wind stress components are:

$$\tau_{xz}^S = -\tau_{\theta z}^S \sin \theta = -\frac{\rho T}{2} r \sin \theta = -\frac{\rho T}{2} y, \quad (7.5.8)$$

$$\tau_{yz}^S = \tau_{\theta z}^S \cos \theta = \frac{\rho T}{2} r \cos \theta = \frac{\rho T}{2} x, \quad (7.5.9)$$

Kinematically we assume that

$$w = 0, \quad z = H. \quad (7.5.10)$$

7.5.1 Inviscid core

Outside the surface and bottom boundary layers, we have

$$-f v_I = -\frac{1}{\rho} \frac{\partial p}{\partial x} \quad (7.5.11)$$

$$f u_I = -\frac{1}{\rho} \frac{\partial p}{\partial y} \quad (7.5.12)$$

This is clearly the state of geostrophic balance. Momentum balance in the vertical direction is trivial,

$$0 = -\frac{1}{\rho} \frac{\partial p}{\partial z}$$

Consequently u_I and v_I must be independent of z . in accordance with the Taylor-Proudman theorem. Note that conservation of mass is automatically satisfied,

$$\frac{\partial u_I}{\partial x} + \frac{\partial v_I}{\partial y} = 0$$

and the vorticity is

$$\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y} = -\frac{1}{f} \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} \right)$$

The horizontal components $u_I(x, y)$, $v_I(x, y)$ are not determined yet. The vertical velocity w_I can at best be a constant in z .

7.5.2 Bottom boundary layer

Let us keep the dominant viscous stress terms in the momentum equations,

$$-f(v - v_I) = \nu \frac{\partial^2 (u - u_I)}{\partial z^2} \quad (7.5.13)$$

$$f(u - u_I) = \nu \frac{\partial^2 (v - v_I)}{\partial z^2} \quad (7.5.14)$$

The boundary conditions are

$$\begin{aligned} u - u_I = -u_I & \quad v - v_I = -v_I & \quad z = 0 \\ u - u_I \rightarrow 0 & \quad v - v_I \rightarrow 0 & \quad z \gg \delta \end{aligned}$$

where

$$\delta = \sqrt{\frac{2\nu}{f}} \quad (7.5.15)$$

is the Ekman boundary layer thickness.

The solution is left to the reader as an exercise

$$u - u_I = -e^{-z/\delta} \left(u_I \cos \frac{z}{\delta} + v_I \sin \frac{z}{\delta} \right) \quad (7.5.16)$$

$$v - v_I = -e^{-z/\delta} \left(v_I \cos \frac{z}{\delta} - u_I \sin \frac{z}{\delta} \right). \quad (7.5.17)$$

From continuity, the vertical component can be computed. Let $\zeta = z/\delta$,

$$\begin{aligned} \frac{\partial w}{\partial z} &= \frac{1}{\delta} \frac{\partial w}{\partial \zeta} = - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \\ &= \left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y} \right) e^{-\zeta} \sin \zeta + \left(\frac{\partial u_I}{\partial x} + \frac{\partial v_I}{\partial y} \right) (e^{-\zeta} \cos \zeta). \end{aligned} \quad (7.5.18)$$

The second term vanishes, hence,

$$\begin{aligned} w &= \delta \int_0^\zeta d\zeta \left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y} \right) e^{-\zeta} \sin \zeta \\ &= \delta \left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y} \right) \frac{e^{-\zeta}}{2} (-\sin \zeta - \cos \zeta) \Big|_0^\zeta \\ &= \frac{\delta}{2} \left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y} \right) [1 - e^{-\zeta} (\cos \zeta + \sin \zeta)]. \end{aligned}$$

At the outer edge of the bottom boundary layer, $\zeta = z/\delta \gg 1$

$$w(\infty) \equiv \frac{\delta}{2} \left(\frac{\partial v_I}{\partial x} - \frac{\partial u_I}{\partial y} \right) = \frac{\delta}{2} \omega_I \quad (7.5.19)$$

where ω_I is the vorticity in the geostrophic interior. Thus there is vertical flux from the bottom boundary layer when the interior flow is horizontally nonuniform; this is called the Ekman pumping!

We still don't know the geostrophic flow field.

7.5.3 Surface boundary layer

The momentum equations are

$$\begin{aligned} -f(v - v_I) &= \nu \frac{\partial^2 (u - u_I)}{\partial z^2} \\ f(u - u_I) &= \nu \frac{\partial^2 (v - v_I)}{\partial z^2}. \end{aligned} \quad (7.5.20)$$

On $z = H$ the boundary conditions are

$$\nu \frac{\partial u}{\partial z} = -\frac{T}{2}y, \quad \nu \frac{\partial v}{\partial z} = \frac{T}{2}x, \quad z = H \quad (7.5.21)$$

Far beneath the surface

$$u \rightarrow u_I, \quad v \rightarrow v_I; \quad (H - z) \gg \delta \quad (7.5.22)$$

Let us introduce the boundary-layer coordinate

$$\eta = \frac{H - z}{\delta} \quad 0 < \eta < \infty. \quad (7.5.23)$$

so that

$$\frac{\partial}{\partial z} \rightarrow -\frac{1}{\delta} \frac{\partial}{\partial \eta} \quad (7.5.24)$$

The solution satisfies the momentum equations and (7.5.22) is of the form

$$u - u_I = e^{-\eta} (A \cos \eta + B \sin \eta) \quad (7.5.25)$$

$$v - v_I = e^{-\eta} (B \cos \eta - A \sin \eta). \quad (7.5.26)$$

In order to satisfy (7.5.21), we first note that

$$\frac{\partial u}{\partial \eta} = e^{-\eta} ((-A + B) \cos \eta + (-A - B) \sin \eta) \quad (7.5.27)$$

$$\frac{\partial v}{\partial \eta} = e^{-\eta} ((-A - B) \cos \eta + (A - B) \sin \eta). \quad (7.5.28)$$

Applying (7.5.21), we get

$$-\frac{\nu}{\delta}(-A + B) = -\frac{Ty}{2}, \quad -\frac{\nu}{\delta}(-A - B) = \frac{Tx}{2} \quad (7.5.29)$$

with the results,

$$A = \frac{T\delta}{4\nu}(x - y), \quad B = \frac{T\delta}{4\nu}(x + y) \quad (7.5.30)$$

Hence the horizontal velocities are

$$u - u_I = \frac{T\delta}{4\nu} e^{-\eta} ((x - y) \cos \eta + (x + y) \sin \eta) \quad (7.5.31)$$

$$v - v_I = \frac{T\delta}{4\nu} e^{-\eta} ((x + y) \cos \eta - (x - y) \sin \eta). \quad (7.5.32)$$

By continuity

$$\begin{aligned}\frac{\partial w}{\partial z} &= -\frac{1}{\delta} \frac{\partial w}{\partial \eta} = -\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right), \\ \frac{\partial w}{\partial \eta} &= \frac{T\delta}{4\nu} e^{-\eta} (2 \cos \eta + 2 \sin \eta)\end{aligned}$$

the vertical velocity can be found,

$$\begin{aligned}w(\eta) &= \frac{T\delta^2}{2\nu} \int_0^\eta d\eta e^{-\eta} (\cos \eta + \sin \eta) \\ &= \frac{T\delta^2}{4\nu} \left[e^{-\eta} (-\cos \eta + \sin \eta) + e^{-\eta} (-\cos \eta - \sin \eta) \right]_0^\eta \\ &= \frac{T\delta^2}{2\nu} (1 - e^{-\eta} \cos \eta)\end{aligned}\tag{7.5.33}$$

At the outer edge of the surface boundary layer $\eta \gg 1$

$$w(\infty) = w_T = \frac{T\delta^2}{2\nu}\tag{7.5.34}$$

By Taylor-Proudman theorem, $w(z) = w_B = w_T$. Therefore

$$w_B = \frac{\delta^2}{2} \omega_I = \frac{T\delta^2}{2\nu} = w_T\tag{7.5.35}$$

and the interior vorticity is

$$\omega_I = \frac{T\delta}{\nu}.\tag{7.5.36}$$

What are u_I and v_I ? In cylindrical polar coordinates

$$\omega_I = \frac{1}{r} \frac{\partial}{\partial r} (r u_{I\theta}) - \frac{1}{r} \frac{\partial u_{I_r}}{\partial \theta} = \frac{1}{r} \frac{\partial}{\partial r} (r u_{I\theta}).$$

Since $\partial/\partial\theta = 0$, we have ,

$$\begin{aligned}\omega_I &= \frac{1}{r} \frac{d}{dr} (r u_{I\theta}) \\ \frac{d}{dr} (r u_{I\theta}) &= \frac{T\delta}{\nu} r\end{aligned}$$

which implies

$$u_{I\theta} = \frac{T\delta}{2\nu} r.$$

Since

$$\frac{1}{r} \frac{\partial}{\partial r} (r u_{I_r}) + \frac{1}{r} \frac{\partial u_{I\theta}}{\partial \theta} = 0$$

which leads to

$$u_{I_r} = 0.$$

The interior flow is geostrophic and cyclonic.

In cartesian form we have

$$u_I = -u_{I_\theta} \sin \theta = -\frac{T\delta}{2\nu} r \sin \theta, \quad (7.5.37)$$

$$v_I = u_{I_\theta} \cos \theta = \frac{T\delta}{2\nu} r \cos \theta \quad (7.5.38)$$

Now the radial component inside the bottom boundary layer is

$$u_r = u_r - u_{I_r}$$

since $u_{I_r} = 0$. The latter is

$$\begin{aligned} u_r - u_{I_r} &= -e^{-\zeta} [(u_I \cos \zeta + v_I \sin \zeta) \cos \theta + (v_I \cos \zeta - u_I \sin \zeta) \sin \theta] \\ &= -e^{-\zeta} [\cos \zeta (u_I \cos \theta + v_I \sin \theta) + \sin \zeta (v_I \cos \theta - u_I \sin \theta)] \\ &= -e^{-\zeta} \sin \zeta (v_I \cos \theta - u_I \sin \theta) \\ &= -\frac{T\delta r}{2\nu} e^{-\zeta} \sin \zeta (\cos^2 \theta + \sin^2 \theta) \\ &= -\frac{T\delta r}{2\nu} e^{-\zeta} \sin \zeta \end{aligned}$$

and is negative in most of the boundary layer. Hence the flow spirals inward towards the z axis in the bottom boundary layer. Similarly one can show that the flow in the surface boundary layer has an outward radial component.

In summary, the swirling wind induces a vorticity T/ν in the geostrophic interior. The flow in the bottom Ekman layer spirals inward, rises vertically at a uniform velocity while spiralling at the angular velocity T/ν and maintaining a constant vorticity in the geostrophic interior, then spirals outward in the surface Ekman layer. The flow is therefore cyclonic.