Cartesian Tensors

* A true physical law must be free of the frame of reference (ie, should remain valid in any coordinate system)

e.g. $\underset{\sim}{F} = m \underset{\sim}{a}$

$\underset{\sim}{F}$, $\underset{\sim}{a}$ are vectors, m is a scalar

In terms of components,

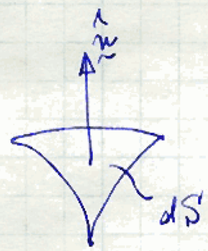
$$F_i = m a_i \quad (i = 1, 2, 3)$$

* Some laws involve tensors.

e.g. $\underset{\sim}{H}_c = [I]_c \underset{\sim}{\omega} \Rightarrow H_c|_i = [I_c]_{ij} \, \omega_j$

H_c, $\underset{\sim}{\omega}$ are vectors, $[I]_c$ is the tensor of inertia of rigid body at CM.

* In continuum mechanics, stress $[\sigma]$ and strain tensor $[e]$

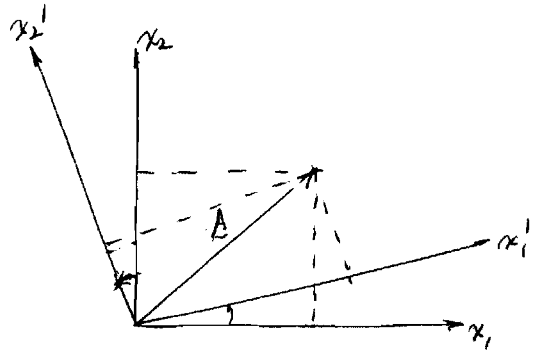


$\underset{\sim}{F} = [\sigma] \hat{\underset{\sim}{n}} \, dS \Rightarrow$

$$F_i = [\sigma]_{ij} \, \hat{n}_j \cdot dS$$

* Physical laws should remain invariant under coordinate transformations (rotations)

Transformation of vectors



$$\underset{\sim}{A} = A_1 \hat{\underset{\sim}{e}}_1 + A_2 \hat{\underset{\sim}{e}}_2 = A_1' \hat{\underset{\sim}{e}}_1' + A_2' \hat{\underset{\sim}{e}}_2'$$

$$\begin{cases} \hat{\underset{\sim}{e}}_1 = \cos(\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_1')\hat{\underset{\sim}{e}}_1' + \cos(\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2')\hat{\underset{\sim}{e}}_2' \\ \hat{\underset{\sim}{e}}_2 = \cos(\hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_1')\hat{\underset{\sim}{e}}_1' + \cos(\hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_2')\hat{\underset{\sim}{e}}_2' \end{cases}$$

$$\therefore \begin{cases} A_1' = A_1 \cos(\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_1') + A_2 \cos(\hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_1') \\ A_2' = A_1 \cos(\hat{\underset{\sim}{e}}_1, \hat{\underset{\sim}{e}}_2') + A_2 \cos(\hat{\underset{\sim}{e}}_2, \hat{\underset{\sim}{e}}_2') \end{cases}$$

Similarly, in 3-D: $\underline{\underline{A_i' = C_{ik} A_k}}$ $\quad C_{ik} = \cos(\hat{\underset{\sim}{e}}_i, \tilde{\underset{\sim}{e}}_k)$

* C_{ik} describes an <u>orthogonal transformation</u> (rotation)

$$|\underset{\sim}{A}| = A_i A_i = A_i' A_i'$$

$$\Rightarrow \quad A_i' A_i' = C_{ik} A_k C_{ij} A_j = C_{ik} C_{ij} A_k A_j = A_j A_j$$

$$\therefore \quad C_{ik} C_{ij} = \begin{cases} 1 & k = j \\ 0 & k \neq j \end{cases} \Rightarrow \underline{\underline{C_{ik} C_{ij} = \delta_{kj}}}$$

In matrix notation, $\underline{\underline{[C]^t [C] = 1}}$ <u>orthogonal matrix</u>

(Only 3 components of C_{ij} are independent)

For an orthogonal matrix $\underline{\underline{[C]^{-1} = [C]^t}}$

Transformation of tensors

* A tensor of rank r, [T], is an array of components

$$T_{\underbrace{ij\ldots m}_{r \text{ indices}}}$$

The components of a tensor transform as follows

$$\underline{\underline{T'_{ijk\ldots m} = C_{is} C_{jt} \cdots C_{mv} T_{st\ldots v}}}$$

* In particular,

for $r = 0 \Rightarrow$ scalar

$r = 1 \Rightarrow$ vector $A'_i = C_{ik} A_k$

For $\underline{r=2}$ (tensor of rank 2) : $\underline{\underline{T'_{ij} = C_{is} C_{jt} T_{st}}}$

e.g. consider $\underline{\underline{H_i = I_{ij} \omega_j}}$

Under rotation of coordinates, since $\underset{\sim}{H}$, $\underset{\sim}{\omega}$ are vectors,

$$\left.\begin{array}{l} H'_k = C_{ki} H_i \Rightarrow H_i = C_{ki} H'_k \\ \omega'_\ell = C_{\ell j} \omega_j \Rightarrow \omega_j = C_{\ell j} \omega'_\ell \end{array}\right\}$$

$\therefore \quad H_i = I_{ij} \omega_j \Rightarrow C_{ki} H'_k = I_{ij} C_{\ell j} \omega'_\ell$

Using the fact that $C_{mi} C_{ki} = \delta_{mk}$,

$$\underbrace{C_{mi} C_{ki}}_{\delta_{mk}} H'_k = C_{mi} I_{ij} C_{\ell j} \omega'_\ell \Rightarrow H'_m = \underbrace{C_{mi} C_{\ell j} I_{ij}}_{I'_{m\ell}} \omega'_\ell$$

$$\Rightarrow \underline{\underline{H'_m = I'_{m\ell} \omega'_\ell}}$$

STAEDTLER® No. 937 811E Engineer's Computation Pad

* Addition of tensors

If A_{ij}, B_{ij} are tensors, then $E_{ij} = A_{ij} + B_{ij}$ is also a tensor

* Multiplication of tensors

$$\underbrace{E_{ij\ldots k rs\ldots t}}_{b+c} = A_{\underbrace{ij\ldots k}_{b}} B_{\underbrace{rs\ldots t}_{c}}$$

e.g. $E_{ijrs} = A_{ij} B_{rs}$

Consider $E'_{ijrs} \equiv A'_{ij} B'_{rs}$; is this a tensor?

$$E'_{ijrs} = A'_{ij} B'_{rs} = \left(C_{ik} C_{jl} A_{kl}\right)\left(C_{rm} C_{sn} B_{mn}\right)$$

$$= C_{ik} C_{jl} C_{rm} C_{sn} \underbrace{A_{kl} B_{mn}}_{E_{klmn}}$$

* Contraction

Summing over two equal indices $\Rightarrow$ tensor $r-2$

e.g. $A'_{rssu} = C_{ri} \underbrace{\left(C_{sj} C_{sk}\right)}_{\delta_{jk}} C_{ul} A_{ijkl}$

$\Rightarrow A'_{rssu} = C_{ri} C_{ul} A_{ijjl}$ is a $r=2$ tensor

e.g. $A_i B_j = T_{ij} \Rightarrow A_i B_i = \underset{\sim}{A} \cdot \underset{\sim}{B}$ scalar product

STAEDTLER® No. 937 811E Engineer's Computation Pad

* <u>Gradient of a tensor</u>

In general, $\underbrace{T_{ij\ldots kl}}_{r+1} = \dfrac{\partial R_{\overbrace{ij\ldots k}^{r}}}{\partial x_l}$ is a tensor of rank $r+1$

e.g., the gradient of a scalar is a vector:

$$V_i = \frac{\partial \sigma}{\partial x_i}$$

$$V_i' = \frac{\partial \sigma}{\partial x_i'} = \frac{\partial \sigma}{\partial x_j}\underbrace{\frac{\partial x_j}{\partial x_i'}}_{C_{ij}} \qquad \text{But } x_i' = C_{ik}x_k \Rightarrow x_j = C_{kj}x_k'$$

$$\therefore \quad V_i' = \frac{\partial \sigma}{\partial x_j}C_{ij} = C_{ij}V_j$$

* <u>divergence</u> is combination of gradient + contraction

In general, $T_{ij\ldots k} = \dfrac{\partial R_{ij\ldots kl}}{\partial x_l}$

e.g. $\nabla \cdot q = \dfrac{\partial q_i}{\partial x_i} = \dfrac{\partial q_j'}{\partial x_j'}$ (scalar)

$$\frac{\partial q_j'}{\partial x_j'} = \frac{\partial}{\partial x_j'}C_{js}q_s = C_{js}\frac{\partial q_s}{\partial x_i}\underbrace{\frac{\partial x_i}{\partial x_j'}}_{C_{ji}} = \underbrace{C_{js}C_{ji}}_{\delta_{si}}\frac{\partial q_s}{\partial x_i} = \frac{\partial q_i}{\partial x_i}$$

* The <u>strain-rate</u> $e_{ij} = \dfrac{\partial q_i}{\partial x_j} + \dfrac{\partial q_j}{\partial x_i}$

<u>rotation</u> $\Omega_{ij} = \dfrac{\partial q_j}{\partial x_i} - \dfrac{\partial q_i}{\partial x_j}$

are <u>tensors</u>