10.302
Recitation # 11 , 12/4/01
MJL

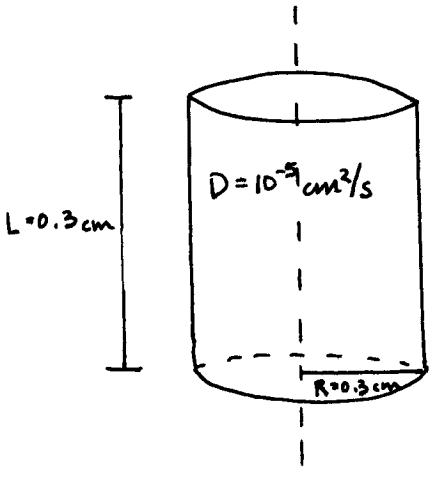
Problem 1

A porous cylindrical pellet 0.3 cm in diameter and 0.3 cm high is introduced into a bath containing a dilute solution (10^{-5} gm moles/cc) of a platinum salt and is maintained in the bath for a given period of time.

You have been asked to estimate the uniformity of the distribution of salt which results from the process. For this purpose, please evaluate the concentration at the center of one of the planar faces, at the geometric center of the pellet, and at the edge where the planar face and the cylindrical surface intersect. In all cases, express your answer relative to the initial concentration. Consider immersion times of 100 seconds and 1000 seconds.

Data

- the effective diffusivity of the platinum salt in the pellet is 10^{-5} cm^2/sec
- the external mass transfer coefficient, hm, to all faces of the pellet is 1.5×10^{-4} cm/sec.



$C_\infty = 10^{-5}$ gm moles/cc

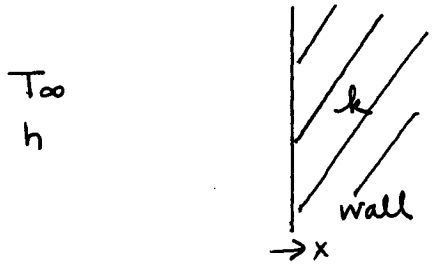
$h_m = 1.5 \times 10^{-4}$ cm/s

- transient
- two spatial dimensions
- dilute
- no reactions
- $S_{1/2} = 1$

Problem 1 continued

Recall all our solution strategies for transient heat transfer (all conduction) problems. The first step was always to evaluate the Biot number (Bi). A Biot number exists in diffusive mass transfer as well (Bi_m). This Bi_m expresses the ratio of internal resistance to mass transfer to external resistance.

Heat transfer



boundary condition at interface:

$$-k\frac{dT}{dx}(x=0) = -h\left(T(0) - T_\infty\right)$$

$$\theta = T - T_\infty, \quad \eta = x/L$$

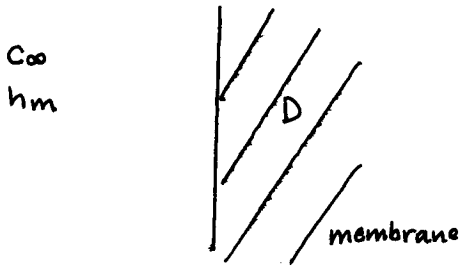
$$-k\frac{d\theta}{d\eta} = -hL\theta$$

$$\frac{d\theta}{d\eta} = +Bi\,\theta, \quad Bi = \frac{hL}{k}$$

h [=] W/m².K

k [=] W/m.K

Mass transfer



boundary condition at interface:

$$-D\frac{dc_M}{dx}(x=0) = -h_m\left(c(0) - c_\infty\right)$$

$$-D\frac{dc_M}{dx}(x=0) = -\frac{h_m}{s}\left(sc(0) - sc_\infty\right)$$

$$-D\frac{dc_M}{dx}(x=0) = -\frac{h_m}{s}\left(c_M(0) - sc_\infty\right)$$

$$\theta = c_M - sc_\infty, \quad \eta = x/L$$

$$-D\frac{d\theta}{d\eta} = -\frac{h_m L}{s}\theta$$

$$\frac{d\theta}{d\eta} = +\frac{h_m L}{sD}\theta = +Bi_m\theta$$

h_m [=] m/s

By analogy with HT:

if $Bi \ll 1$ lumped capacitance is OK

if $Bi \geq 1$ internal spatial effects are important

Problem 1 continued

For our problem,

$$Bi_m = \frac{h_m L}{\mathcal{D}S} = \frac{(1.5\times10^{-4}\ cm/s)(0.15\ cm)}{10^{-5}\ cm^2/s} = 2.25 > 0.1$$

Thus, we cannot use a "lumped capacitance" approach. What approaches should we think about? Can we treat the pellet as a semi-infinite solid? How far has the mass transfer wave propagated in to the pellet at times of interest? To answer these questions, we need to calculate the Fourier number for this diffusive process.

By analogy to HT, where $Fo = \alpha t/L^2$, the Fourier number for mass transfer is $Fo = \mathcal{D}t/L^2$.

t (s)	Fo	
100	$(10^{-5})(100)/(0.15)^2 = 0.044$	→ for this t, can be treated as semi-infinite
1000	$(10^{-5})(1000)/(0.15)^2 = 0.444$	→ here a first term S.O.V. solution is valid

However, this problem also involves two spatial coordinates. How do we treat this? Go back to §5.8 on multi dimensional effects in HT. For two-dimensional, transient conduction in a short cylinder,

$$\frac{T(r,x,t)-T_\infty}{T_i-T_\infty} = \left.\frac{T(x,t)-T_\infty}{T_i-T_\infty}\right|_{\text{plane wall}} \cdot \left.\frac{T(r,t)-T_\infty}{T_i-T_\infty}\right|_{\text{infinite cylinder}}$$

Similarly for mass transfer

$$\frac{c(r,z,t)-c_\infty}{c_i-c_\infty} = \left.\frac{c(z,t)-c_\infty}{c_i-c_\infty}\right|_{\text{infinite cylinder}} \cdot \left.\frac{c(x,t)-c_\infty}{c_i-c_\infty}\right|_{\text{plane wall}}$$

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Problem 1 continued

. For $t = 1000$ s ($Bi = 0.444$), the first term approximate solution from a separation of variables solution is sufficient.

plane wall: $\theta^* = \frac{c - c_\infty}{c_i - c_\infty} = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) \cos(\zeta_n x^*)$ full solution

θ^* (first term) $= C_1 \exp(-\zeta_1^2 Fo) \cos(\zeta_1 x^*)$ approx. ($Fo > 0.2$)

infinite cylinder: $\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) J_0(\zeta_n r^*)$ exact solution

$\theta^* = C_1 \exp(-\zeta_1^2 Fo) J_0(\zeta_1 r^*)$ approx. ($Fo > 0.2$)

So, for $t = 1000$ s,

$Bi = 2.25 \rightarrow$ Table 5.1

	ζ_1	C_1
Plane Wall	1.1058	1.1892
Inf Cyl.	1.6468	1.3586

①

Center of a planar face (0, 0.15, 1000)

$\theta^*|_{\text{plane wall}} = (1.1892) \exp(-1.1058^2 \cdot 0.444) \cos(1.1058 \cdot 1) = 0.3093$

$\theta^*|_{\text{inf cyl}} = (1.3586) \exp(-1.6468^2 \cdot 0.444) J_0(1.6468 \cdot 0) = 0.4075$

$\theta^* = (0.3093)(0.4075) = 0.126$

$\frac{c - c_\infty}{c_i - c_\infty} = 0.126 \rightarrow c(0, 0.15, 1000) = 0.874\, c_\infty$

Problem 1 continued

② geometric center of the pellet $(0, 0, 1000)$

$\theta^*|_{\text{plane wall}} = 1.1872 \exp(-1.1058^2 \cdot 0.444) \cos(1.1058 \cdot 0) = 0.6898$

$\theta^*|_{\text{inf cyl}} = 0.4075$

$\theta^* = (0.6898)(0.4075) = 0.281$

$C(0,0,1000) = 0.719\ C_\infty$

③ at the edge of planar face and cyl. wall $(0.15, 0.15, 1000)$

$\theta^*|_{\text{plane wall}} = 1.1872 \exp(-1.1058^2 \cdot 0.444) \cos(1.1058 \cdot 1) = 0.3093$

$\theta^*|_{\text{inf cyl}} = 1.3586 \exp(-1.6468^2 \cdot 0.444) \underbrace{J_0(1.648 \cdot 1)}_{0.4278} = 0.1743$

$\theta^* = (0.3093)(0.1743) = 0.054$

$C(0.15, 0.15, 1000) = 0.946\ C_\infty$

Summary for $t = 1000s$

geometric center	$(0,0,1000)$	$0.719\ C_\infty$
center of planar face	$(0, 0.15, 1000)$	$0.894\ C_\infty$
edge of face & wall	$(0.15, 0.15, 1000)$	$0.946\ C_\infty$

How do we address the shorter time ($Fo = 0.044$, $t = 100\,s$)?

One way is to use the same overall solution strategy and evaluate each piece (plane wall & inf cyl) using the exact solution from S.O.V. Another way would be to use a "semi-infinite solid" approach (similarity solution) for the two pieces (inf cyl & plane wall).

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Problem 1 Continued

Plane Wall: $$\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) \cos(\zeta_n x^*) \quad [5.39a]$$

$$C_n = \frac{4 \sin \zeta_n}{2\zeta_n + \sin(2\zeta_n)} \quad [5.39b]$$

$$\zeta_n \tan \zeta_n = Bi \quad [5.39c]$$

Inf Cyl: $$\theta^* = \sum_{n=1}^{\infty} C_n \exp(-\zeta_n^2 Fo) J_0(\zeta_n r^*) \quad [5.47a]$$

$$C_n = \frac{2}{\zeta_n} \frac{J_1(\zeta_n)}{J_0^2(\zeta_n) + J_1^2(\zeta_n)} \quad [5.47b]$$

$$\zeta_n \frac{J_1(\zeta_n)}{J_0(\zeta_n)} = Bi \quad [5.47c]$$

plane		inf cyl	
ζ_n	C_n	ζ_n	C_n
1.1058	1.1865	1.6468	1.3583
3.69	-0.2522	4.34	-0.5339

		θ^*		
		plane	inf cyl	C
geom. center	(0, 0, 100*) = (0, 0, 0.044)	0.9859	0.9724	$0.04 C_\infty$
center of face	(0, 0.15, 100) = (0, 1, 0.044)	0.6224	0.9724	$0.39 C_\infty$
edge of face & wall	(0.15, 0.15, 100) = (1, 1, 0.044	0.6224	0.5992	$0.63 C_\infty$

Note: Only two terms were used here because it was found that the third and higher order terms contributed to a negligible extent, even at $Fo = 0.044$. Note that this method could also be used by using graphical results for the exact S.O.V. solution.

Note on using a similarity solution: As mentioned earlier, this problem could also be solved by using similarity solutions for the inf cyl and plane wall. In fact, since we are at such low Fo, we expect that the penetration depth will be so small that the inf cyl problem will be well approximated by a plane wall solution as well. Thus Eq. 5.60 can be used for both pieces. The results achieved thusly compare well with the above results.

Problem 2

Long polymeric cylinders with a radius of 2 mm have been produced in the laboratory with the intent of using them as part of an implantable biomedical device. Unfortunately, the rods contain a small amount of monomeric residue which must be extracted prior to use. In order to achieve this, the rods will be immersed in a large volume of well-stirred solvent. You have been asked to estimate the time required to achieve a ten-fold reduction in the monomer concentration at the centerline of the cylinders. The diffusivity of the monomer within the polymer is 10^{-6} cm²/s, and the mass transfer coefficient between the surface of the rods and the solvent is 10^{-3} cm/s. Two solvents are under consideration and the solubilities of the monomer in the two solvents are known to be quite different. Specifically, the equilibrium partitioning of the monomer between the solvent and the polymer is given by:

solvent A : $[C_M]_A = [C_M]_P \rightarrow S_{P/A} = 1 = S_{A/P}$

solvent B : $[C_M]_B = 0.05\,[C_M]_P \rightarrow$ ~~$S_{P/A} = 0.05$~~ $S_{B/P} = 0.05$, $S_{P/B} = 20$

where, $[C_M]$ is the monomer concentration, A refers to solvent A, B refers to solvent B, P refers to polymer.

a) What is the Biot number associated with the use of solvent A?

$$Bi_m = \frac{h_m L}{S \mathcal{D}} = \frac{(10^{-3}\ \text{cm/s})(0.2\ \text{cm})}{1\ (10^{-6}\ \text{cm}^2/\text{s})} = 200$$

b) What is the Biot number associated with solvent B?

$$Bi_m = \frac{h_m L}{S_{P/B} \mathcal{D}} = \frac{(10^{-3}\ \text{cm/s})(0.2\ \text{cm})}{20\ (10^{-6}\ \text{cm}^2/\text{s})} = 10$$

Problem 2 continued

c) If solvent A is used, how many hours will be required to achieve a 10-fold reduction in the centerline concentration?

Here, we will use Fig. P.4 on page 950 and solve the problem by analogy to heat transfer. Since $c_{M,A}(x=\infty) = 0$, we want $\theta_0^* = \frac{c(0) - c_\infty}{c_i - c_\infty} = 0.1 \rightarrow c(0) = 0.1\, c_i$.

$$Bi = 200 \rightarrow Bi^{-1} = 0.005 \rightarrow \text{for } \theta_0^* = 0.1 \quad F_0 = 0.5$$

$$F_0 = \frac{\mathcal{D}t}{L^2} \rightarrow t = \frac{F_0 L^2}{\mathcal{D}} = \frac{(0.5)(0.2\ cm)^2}{10^{-6}\ cm^2/s}$$

$$t = 5.56\ hr.$$

d) If solvent B is used, how many hours will it take to achieve a 10-fold reduction in the centerline concentration?

Again, $\theta_0^* = 0.1$, but $Bi^{-1} = \frac{1}{10} = 0.1 \rightarrow F_0 = 0.65$

$$t = \frac{F_0 L^2}{\mathcal{D}} = \frac{(0.65)(0.2\ cm)^2}{10^{-6}\ cm^2/s} = 7.22\ hr.$$

Note: this could have also been done using the one-term approximate solutions from S.O.V. and finding the F_0's