**13.012 READING 6:**
**LINEAR FREE SURFACE WAVES**

FALL 2003



## 1. Free Surface Water Waves

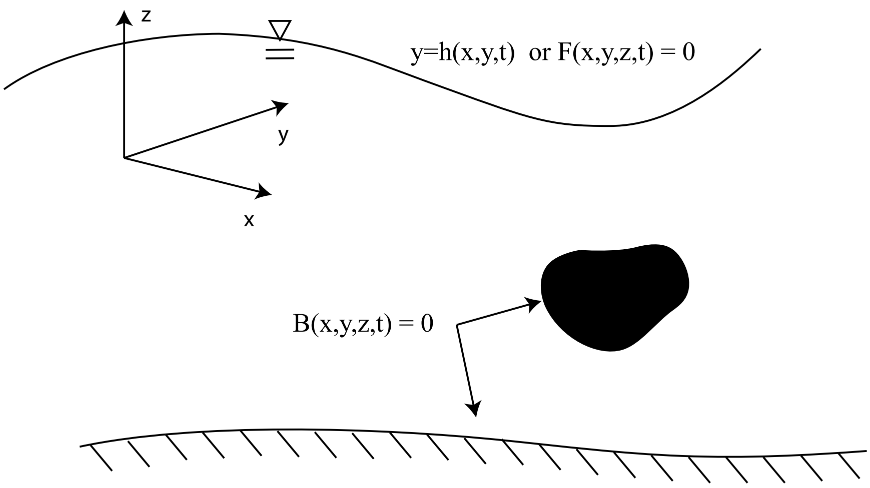


Figure 1. Free surface water wave problem.

In order to determine an exact equation for the problem of free surface gravity waves we will assume potential theory (ideal flow) and ignore the effects of viscosity. Waves in the ocean are not typically uni-directional, but often approach structures from many directions. This complicates the problem of free surface wave analysis, but can be overcome through a series of assumptions.

To setup the exact solution to the free surface gravity wave problem we first specify our unknowns:

- Velocity Field: $\overrightarrow{v}(x, y, z, t) = \nabla\phi(x, y, z, t)$
- Free surface elevation: $\eta(x, y, t)$ or $F(x, y, z, t)$
- Pressure $p(x, y, z, t)$

Next we need to set up the equations and conditions that govern the problem:

- Continuity (Conservation of Mass): $\nabla^2\phi = 0$ for $y < \eta$ or $F < 0$ (Laplace's Equation)
- Bernoulli's Equation (given some $\phi$): $\frac{\partial\nabla}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2 + \frac{p-p_a}{\rho} + gz = 0$ for $y < \eta$ or $F < 0$
- No disturbance far away: $\frac{\partial\nabla}{\partial t}$, $\nabla\phi \rightarrow 0$ and $p = p_a - \rho g z$

Finally we need to dictate the boundary conditions at the free surface, seafloor and on any body in the water:

**(1) Pressure is constant across the free surface interface:** $p = p_{atm}$ **on** $z = \eta$**.**

$$p = -\rho\frac{\partial\phi}{\partial t} - \frac{1}{2}V^2 - gz + c(t) = p_{atm}. \tag{1.1}$$

Choosing a suitable integration constant, $c(t) = p_{atm}$, the boundary condition on $z = \eta$ becomes

$$\rho\{\frac{\partial\phi}{\partial t} + \frac{1}{2}V^2 + g\eta\} = 0. \tag{1.2}$$

**(2) Once a particle is on the free surface, it remains there always.** Similarly, the normal velocity of a particle on the surface follows the normal velocity of the surface itself.

$$z_p = \eta(x_p, t) \tag{1.3}$$

$$z_p + \delta z_p = \eta(x_p + \delta x_p, t + \delta t) = \eta(x_p, t) + \frac{\partial\eta}{\partial x}\delta x_p + \frac{\partial\eta}{\partial t}\delta t \tag{1.4}$$

On the surface, where $z_p = \eta$, we can reduce the above equation to

$$\delta z_p = \frac{\partial\eta}{\partial t}u\ \delta t + \frac{\partial\eta}{\partial t}\ \delta t \tag{1.5}$$

and substitute $\delta z_p = w\ \delta t$ and $\delta x_p = u\ \delta t$ to show that the normal velocity follows the particle:

$$w = u\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial t}\ on\ z\ =\ \eta. \tag{1.6}$$

**(3) On an impervious boundary $B(x, y, z, t) = 0$.** Velocity of the fluid normal to the body must be equal to the body velocity in that direction:

$$\overrightarrow{v} \cdot \hat{n} = \nabla\phi \cdot \hat{n} = \frac{\partial \phi}{\partial n} = \overrightarrow{U}(\overrightarrow{x}, t) \cdot \hat{n}(\overrightarrow{x}, t) = U_n\ on\ B = 0. \tag{1.7}$$

Alternately a particle P on $B$ remains on $B$ always; ie. $B$ is a material surface. For example: if P is on $B$ at some time $t = t_o$ such that

$$B(\overrightarrow{x}, t_o) = 0,\ then\ B(\overrightarrow{x}, t_o) = 0\ for\ all\ t, \tag{1.8}$$

so that if we were to follow P then $B = 0$ always. Therefore:

$$\frac{DB}{Dt} = \frac{\partial B}{\partial t} + (\nabla\phi \cdot \nabla)B = 0\ on\ B = 0. \tag{1.9}$$

Take for example a flat bottom at $z = -h$:

$$\partial\phi/\partial y = 0 \text{ on } z = -h$$

## 2. Linear Waves

To simplify the complex problem of ocean waves we will consider only small amplitude waves (such that the slope of the free surface is small). This means that the wave amplitude is much smaller than the wavelength of the waves.

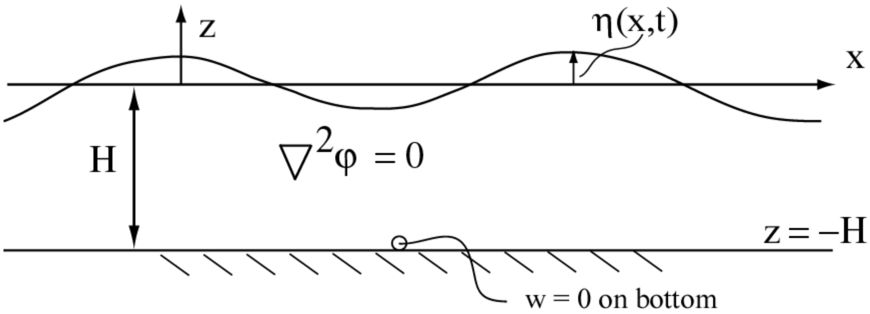


FIGURE 2. Linearized Wave Problem.

As a general rule of thumb for wave height to wavelength ratios less than 1/7 we can linearize the free surface boundary conditions. Non-dimensional variables can be used to assess which terms can be dropped.

| | | |
|---|---|---|
| | $\eta = a\ \eta^*$ | $\omega\ t = t^*$ |
| Non-dimensional variables: | $u = a\omega\ u^*$ | $x = \lambda\ x^*$ |
| | $w = a\omega\ w^*$ | $\phi = a\omega\lambda\ \phi^*$ |

Looking back to equation 1.1 we can evaluate the relative magnitude of each term by plugging in the non-dimensional versions of each variable. For example:

$d\phi = a\omega\lambda\ d\phi^*$

$dt = 1/\omega\ dt^*$

$dx = \lambda\ dx^*$

Let's compare $\frac{\partial\phi}{\partial t}$ and $(\frac{\partial\phi}{\partial x})^2$ to determine which terms can be dropped from equation 1.1.

$$\frac{(\frac{\partial\phi}{\partial x})^2}{\frac{\partial\phi}{\partial t}} = \frac{a^2\omega^2}{a\omega^2\lambda}\ \frac{(\frac{\partial\phi^*}{\partial x^*})}{\frac{\partial\phi^*}{\partial t^*}} = \frac{a}{\lambda}\ \frac{(\frac{\partial\phi^*}{\partial x^*})}{\frac{\partial\phi^*}{\partial t^*}} \tag{2.1}$$

Here we see that since $h/\lambda << 1/7$, then

$$(\frac{\partial\phi}{\partial x})^2 << \frac{\partial\phi}{\partial t}, \tag{2.2}$$

and we can drop the smaller term resulting in linearized boundary conditions

$$\left.\begin{aligned}\frac{\partial \eta}{\partial t} &= \frac{\partial \phi}{\partial z}\\ \frac{\partial \phi}{\partial t} + g\eta &= 0\end{aligned}\right\} \text{on } z = \eta$$

Throughout this discussion we have assumed that the wave height is small compared to the wavelength. Along these lines we can also see why $\eta$ is also quite small. We can expand $\phi$ about $z = 0$:

$$\phi(x, z = \eta, t) = \phi(x, 0, t) + \frac{\partial \phi}{\partial z}\ \eta + ... \tag{2.3}$$

The second term, $\frac{\partial \phi}{\partial z}\ \eta$ and the subsequent higher order terms are very small and can be ignored in our linear equations allowing us to rewrite the boundary conditions at $z = \eta$ as boundary conditions on $z = 0$.

$$\boxed{\left.\begin{aligned}\frac{\partial^2 \phi}{\partial t^2} + g\ \frac{\partial \phi}{\partial z} &= 0\\ \eta = -\frac{1}{g}\ \frac{\partial \phi}{\partial t}\end{aligned}\right. \quad \text{on } z\ =\ 0.}$$

2.1. **A Solution to the Wave Equation.** The complete boundary value wave problem consists of the differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \tag{2.4}$$

with the following boundary conditions:

Bottom Boundary Condition:

$$\frac{\partial \phi}{\partial z} = 0\ on\ z\ =\ -H \tag{2.5}$$

Free Surface Dynamic Boundary Condition (FSDBC):

$$\eta = \frac{1}{g}\ (\frac{\partial \phi}{\partial t})\ on\ z\ =\ 0. \tag{2.6}$$

Free Surface Kinematic Boundary Condition (FSKBC):

$$\frac{\partial^2 \phi}{\partial t^2} + g\ \frac{\partial \phi}{\partial z} = 0 \tag{2.7}$$

Through separation of variables we can solve Laplace's equation for $\eta,\ \phi$.

$$\eta(x,t) = a\ \cos(kx - \omega t + \psi) \tag{2.8}$$

$$\phi(x,z,t) = -\frac{a\omega}{k}\ f(z)\ \sin(kx - \omega t + \psi) \tag{2.9}$$

$$u(x,z,t) = a\omega\ f(z)\ \cos(kx - \omega t + \psi) \tag{2.10}$$

$$w(x,z,t) = -a\omega f_1(z)\ \sin(kx - \omega t + \psi) \tag{2.11}$$

$$f(z) = \frac{\cosh[k(z+H)]}{\sinh(kh)} \tag{2.12}$$

$$f_1(z) = \frac{\sinh[k(z+H)]}{\sinh(kh)} \tag{2.13}$$

$$\omega^2 = gk\ \tanh(kH) \Rightarrow \textit{dispersion relation} \tag{2.14}$$

where a, $\omega$, k, $\psi$ are integration constants with physical interpretation: $a$ is the wave amplitude, $\omega$ the wave frequency, $k$ is the wavenumber ($k = 2\pi/\lambda$), and $\psi$ simply adds a phase shift.

## 3. Dispersion Relationship

The dispersion relationship uniquely relates the wave frequency and wave number given the depth of the water. The chosen potential function, $\phi$, must satisfy the free surface boundary condition (equation 2.7) such that pluggin $\phi$ in to the FSKBC we get:

$$-\omega^2 \cosh kh + gk \sinh kh = 0 \tag{3.1}$$

Resulting in the Dispersion relationship:

$$\omega^2 = gk \tanh kh; \tag{3.2}$$

For deep water where $h \to \inf$ $\tanh kh \to 1$ so that the dispersion relationship in deep water is:

$$\omega^2 = gk \tag{3.3}$$

In general, $k \uparrow$ as $\omega \uparrow$ or $\lambda \uparrow$ as $T \uparrow$.

Phase speed, $C_p$, of a wave (velocity that a wave crest is travelling) is found using this relationship:

$$\tfrac{\lambda}{T} = C_p = \tfrac{\omega}{k} = \sqrt{\tfrac{g}{k} \tanh kh}$$

in general. This simplifies for the case of deep water such that

$$C_p = \sqrt{\tfrac{g}{k}}$$

Solution to the dispersion relationship in general form can be found graphically.

## 4. Pressure

The unsteady Bernoulli's equation can be used to find the pressure due to a wave potential, $\phi$.

$$p = -\rho \frac{\partial \phi}{\partial t} \underbrace{-\frac{1}{2}\rho V^2}_{2^{nd}\text{ order}} \underbrace{-\rho g z}_{\text{hydrostatic}}$$

Dropping the second order term, the dynamic pressure, $p_d$, is equal to

$$p_d(x,z,t) = -\rho \frac{\partial \phi}{\partial t} = \frac{a\omega^2}{k}\rho \, f(z) \, \cos(\omega t - kx - \psi) \tag{4.1}$$

$$= \rho \frac{\omega^2}{k} \, f(z) \, \eta(x,t) \tag{4.2}$$

$$\frac{\omega^2}{k} \, f(z) = g \, \tanh(kH) \frac{\cosh[k(z+H)]}{\sinh(kH)} = g \, \frac{\cosh[k(z+H)]}{\cosh(kH)} \tag{4.3}$$

$$\boxed{\text{Dynamic Pressure} \Rightarrow \; p_d(x,z,t) = \rho g \eta(x,t) \, \frac{\cosh[k(z+H)]}{\cosh(kH)}}$$

## 5. Motion of Fluid Particles

Define motion of fluid particles with horizontal motion, $\zeta_p$ and vertical motion, $\eta_p$.

$$u = \frac{\partial \zeta_p}{\partial t} \quad \text{and} \quad w = \frac{\partial \eta_p}{\partial t}$$

$$\zeta_p(x,z,t) = a \, f(z) \, \sin(kx - \omega t) + \psi) \tag{5.1}$$

$$\eta_p(x,z,t) = a \, f_1(z) \, \cos(\omega t - k - x + \psi) \tag{5.2}$$

$$\left[\frac{\zeta_p}{f(z)}\right]^2 + \left[\frac{\zeta_p}{f_1(z)}\right]^2 = a^2 \tag{5.3}$$

$$\left[\frac{\zeta_p}{a\left(\frac{\cosh k(z+H)}{\sinh kh}\right)}\right]^2 + \left[\frac{\zeta_p}{a\left(\frac{\sinh k(z+H)}{\sinh kh}\right)}\right]^2 = 1 \tag{5.4}$$

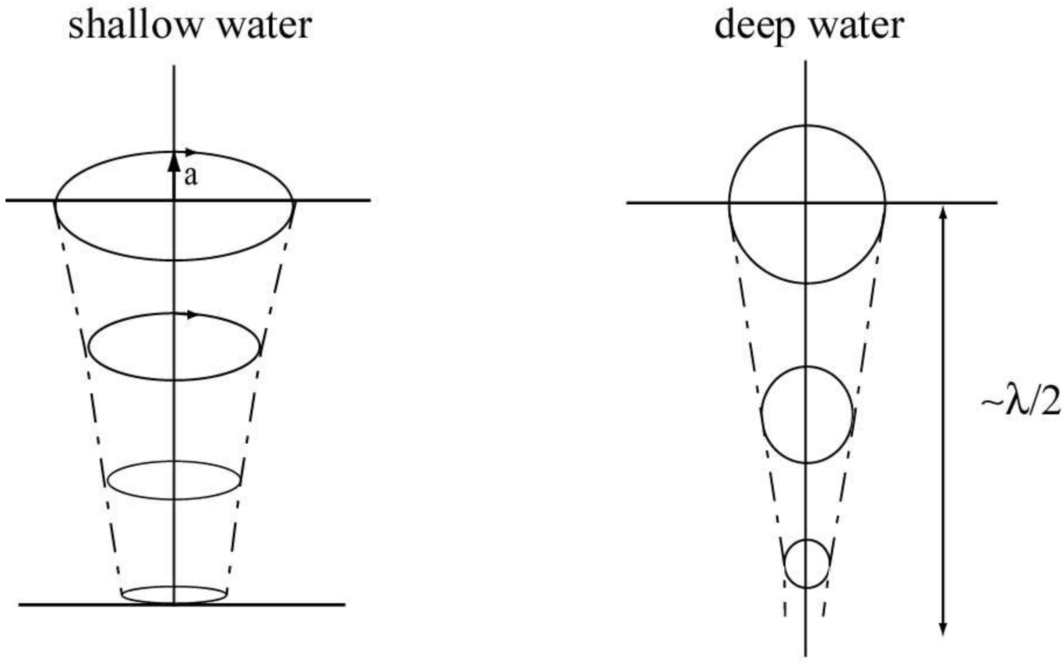


FIGURE 3. Fluid particles move in elliptical orbits below the surface. In deep water, as $H \to \infty$, these ellipese become circular.

5.1. **Deep Water.** In deep water, $H \to \infty$, and $f(z) \approx f_1(z) \approx e^{kz}$. The orbits become circular with exponentially decreasing radius. The particle motion dies out at $z \approx -\lambda/2$.

## 6. Phase and Group Velocity

6.1. **Phase Speed.** The phase speed of a wave is defined as the speed at which the wave is moving. If you were to clock a wave crest you would find that it moves at the phase speed, $C_p = \omega/k$.

6.2. **Group Velocity.** Group velocity is the speed of propagation of a packet, or group, of waves. This is always slower than the phase speed of the waves. In a laboratory setting group speed can be observed by creating a short packet of waves, about 8-10 cycles, and observing this packet as it propagates down a testing tank. The leading edge of the packet will appear to move slower than the waves within the packet. Individual waves will appear at the rear of the packet and propagate to the front, where the pressure forces their apparent disappearance.

We can **derive Group Velocity** starting with a harmonic surface wave with surface elevation

$$\eta(x,t) = a\cos(\omega t - kx) \tag{6.1}$$

and looking at the sum of two cosines with similar frequencies:

$$\begin{aligned}\eta(x,t) &\stackrel{?}{=} \lim_{\delta k,\delta\omega\to 0} \left\{\tfrac{a}{2}\ \cos\left([\omega-\delta\omega]t - [k-\delta k]x\right) + \tfrac{a}{2}\ \cos\left([\omega+\delta\omega]t - [k+\delta k]x\right)\right\} \\ &= \lim_{\delta k,\delta\omega\to 0} \{a\ \cos(\omega t - kx)\cos(\delta\omega t - \delta k x)\}\end{aligned}$$

The envelope (or "Group") travels at:

$$C_g = \frac{\delta\omega}{\delta k} \tag{6.2}$$

In the limit as $\delta\omega$, $\delta k \to 0$, the group velocity becomes

$$C_g = \frac{d\omega}{dk} \tag{6.3}$$

Using the dispersion relation we can derive the group speed.

$$\omega^2 = kg\ \tanh(kH) \tag{6.4}$$

$$2\omega\frac{d\omega}{dk} = g\ \tanh(kH) + \frac{kgH}{\cosh^2(kh)} \tag{6.5}$$

$$C_g = \frac{1}{2}C_p\left\{1 + \frac{kH}{\sinh\ kH\ \cosh\ kH}\right\} \tag{6.6}$$

**Deep Water:** $H \to \infty$

$$\boxed{\omega^2 = kg \text{ and } C_g = \tfrac{1}{2}C_p}$$

**Shallow Water:** $H \to 0$

$$\boxed{\omega = \sqrt{gH}k \text{ and } C_g = C_p}$$

## 7. Wave Energy

**Potential Energy** Only exists at the free surface!

$$E_P = \tfrac{1}{\lambda}\int_0^\lambda \tfrac{1}{2}\rho g\eta^2\,dx = \frac{1}{\lambda}\int_0^\lambda \frac{1}{2}\rho g a^2\cos^2(\omega t - kx)$$

$$E_P = \frac{1}{4}\rho g a^2 \tag{7.1}$$

**Kinetic Energy**

$$E_k = \frac{1}{\lambda}\int_0^\lambda \int_{-H}^0 \frac{1}{2}\rho(u^2 + w^2)\,dz\,dx$$

$$E_k = \frac{1}{4}\rho g a^2 \tag{7.2}$$

**Total Energy per wavelength**

$$\overline{E} = E_P + E_k = \frac{1}{2}\rho g a^2 \tag{7.3}$$

### 7.1. Flux of Energy through a Vertical Plane.

$Power = force * velocity = (p\,dz)u$ Energy flux:

$$\frac{dE}{dt} = \int_{-H}^0 p\ u\ dz \tag{7.4}$$

$$= \frac{1}{2k}\rho g a^2\omega\cos^2(\omega t - kx) \tag{7.5}$$

Average energy flux over one cycle, assuming deep water,

$$\frac{d\overline{E}}{dt} = \frac{1}{T}\int_0^T \frac{dE}{dt}dt = \frac{1}{4k}\rho g a^2\omega = \overline{E}\cdot C_g \tag{7.6}$$

## 8. Useful References

The material covered in this section should be review. If you have not taken 13.021 or a similar class dealing with basic fluid mechanics and water waves please contact the instructor. The references below are merely suggestions for further reading and reference.


- J. N. Newman (1977) *Marine Hydrodynamics* MIT Press, Cambridge, MA.
- O. M. Faltinsen (1990) *Sea Loads on Ships and Offshore Structures* Cambridge University Press, Cambridge, UK.
- M. Rahman (1995) *Water Waves: Relating modern theory to advanced engineering practice* Clarendon Press, Oxford.