

14.06 Problem Set 3 2005 - Solutions

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Question 1. Decentralized Ramsey Model with Labor and Capital Tax

This model is a variation on the Taxation and Redistribution model discussed in the class notes (Chapter 3.5.2). As usual, households maximize the following utility function

$$\int_{t=0}^{\infty} e^{-\rho t} \frac{c^j(t)^{1-\gamma}}{1-\gamma} dt$$

Note that this specification of the utility function assumes that labor supply is exogenous. More specifically, we assume that every household supplies one unit of labor. There is no population growth and no technological progress.

We now introduce a government into the economy, where the government imposes a tax on households' labor income and a separate tax on their capital income. The government then redistributes the tax revenue uniformly across households in the form of a lump sum transfer, $T(t)$. The household budget constraint is

$$\dot{k}^j(t) = (1 - \tau^l)w(t) + (1 - \tau^k)r(t)k^j(t) - c^j(t) - \delta k^j(t) + T(t)$$

The budget constraint of the government is

$$T(t) = \int_j [\tau^l w(t) + \tau^k r(t)k^j(t)] dj$$

a) What is the resource constraint of the economy?

The resource constraint is

$$\dot{k}(t) = f(k(t)) - c(t) - \delta k(t)$$

b) Solve the household's maximization problem and give the system of differential equations that characterizes the optimal solution.

Set up the Hamiltonian as follows

$$H = e^{-\rho t} \frac{c(t)^{1-\gamma}}{1-\gamma} + \mu(t)[(1 - \tau^l)w(t) + (1 - \tau^k)r(t)k(t) - c(t) - \delta k(t) + T(t)]$$

Note that we will drop the subscript j from now on since the problem is the same for all households. The FOCs are

$$\begin{aligned} H_c &= e^{-\rho t} c(t)^{-\gamma} - \mu(t) = 0 \\ H_k &= \mu(t)[(1 - \tau^k)r(t) - \delta] = -\dot{\mu}(t) \end{aligned}$$

Now use the FOC for consumption to find $\mu(t)$ and $\dot{\mu}(t)$

$$\begin{aligned} \mu(t) &= e^{-\rho t} c(t)^{-\gamma} \\ \dot{\mu}(t) &= -\rho e^{-\rho t} c(t)^{-\gamma} - \gamma e^{-\rho t} c(t)^{-\gamma} \frac{\dot{c}(t)}{c(t)} \end{aligned}$$

Plugging these into the FOC for capital gives the Euler equation, which we combine with the resource constraint to get the system of differential equations that characterize the optimal solution. We also need to replace the interest rate in the Euler equation with the marginal product of capital since we know from the firm's maximization problem that $r(t) = f'(k(t))$.

$$\begin{aligned}\frac{\dot{c}(t)}{c(t)} &= \frac{1}{\gamma} [(1 - \tau^k) f'(k(t)) - \delta - \rho] \\ \dot{k}(t) &= f(k(t)) - c(t) - \delta k(t)\end{aligned}$$

c) Draw the Phase diagram. How does it compare to the Phase diagram in the model without distortive taxation?

The $\dot{c} = 0$ locus and the $\dot{k} = 0$ locus are given by

$$\begin{aligned}\dot{c} = 0 &\iff (1 - \tau^k) f'(k^*) - \delta = \rho \\ \dot{k} = 0 &\iff c(t) = f(k(t)) - \delta k(t)\end{aligned}$$

So, the Phase diagram is as usual, except that the $\dot{c} = 0$ locus is further to the left than it would be in an economy without distortive taxation. Note that having a tax on capital decreases the private return to capital (households now get only $(1 - \tau^k)r(t)$ instead of $r(t)$), which implies that households accumulate less capital. Another way to think about this is that the tax on capital lowers the cost of consumption today in terms of consumption tomorrow. This leads agents to consume more initially, so that they accumulate less capital, which implies a lower growth rate of consumption and lower consumption and capital in steady state. This is the substitution effect at work. However, there is no wealth effect in this particular setup since the government gives all the tax revenues back to households.

d) Suppose that the government decides to increase the tax on capital permanently (this is unanticipated). How does this affect the $\dot{c} = 0$ locus, the $\dot{k} = 0$ locus, and the steady state? What happens to c , k , and y ?

The increase in τ^k shifts the $\dot{c} = 0$ locus to the left. The $\dot{k} = 0$ locus does not change. Thus, the economy moves to a lower steady state with lower levels of c and k .

At the time of the tax increase, consumption jumps up onto the new saddle path, which in this case lies above the old steady state (since there is only a substitution effect, consumption today has to go up when the price of consumption today decreases). From then on, consumption decreases, following the new saddle path. The capital stock declines gradually to the new steady state level, and so does output.

e) Now suppose that instead the government decides to increase the tax on labor income. Does this have the same effect as increasing the tax on capital? Why?

The increase in the tax on labor has no effect. Since households are not maximizing over their labor supply, the tax on labor income does not enter into the FOCs and has no distortive effect on the optimal allocation.

f) Finally, suppose that labor supply is endogenous in the maximization problem. Would this change your answer to e) and why? [You do not have to do the math, just give the intuition in one or two sentences.]

In this case, the tax on labor income would affect the optimal allocation similarly to the tax on capital. The tax on labor income would now distort the household's decisions since it lowers the private returns to labor. This implies that households would chose to work less, that it they would consume more leisure, which lead to less output and less investment.

Question 2. AK Model

The discrete time version of this model is discussed in the class notes (Chapter 6.1).

Consider the social planner's problem where utility is given by

$$\int_{t=0}^{\infty} e^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta} dt$$

The resource constraint is

$$\dot{k}(t) = f(k(t)) - c(t) - \delta k(t)$$

and the production function is of the AK form

$$y(t) = Ak(t)$$

where $A > 0$.

a) Solve the social planner's maximization problem and give an expression for optimal consumption growth. Which assumption do we need to impose to ensure perpetual growth?

Set up the Hamiltonian

$$H = e^{-\rho t} \frac{c(t)^{1-\theta}}{1-\theta} + \mu(t)[Ak(t) - c(t) - \delta k(t)]$$

The FOCs are

$$H_c = e^{-\rho t} c(t)^{-\theta} - \mu(t) = 0$$

$$H_k = \mu(t)[A - \delta] = -\dot{\mu}(t)$$

As in Question 1, use the FOC for consumption to find $\mu(t)$ and $\dot{\mu}(t)$ and plug into the FOC with respect to capital. This gives the Euler equation

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta}[A - \delta - \rho]$$

which is the expression for optimal consumption growth. To ensure that growth is always positive, we need to assume

$$A - \delta - \rho > 0$$

b) As in the lecture notes, use the guess that consumption is a linear function of capital

$$c(t) = (1-s)Ak(t).$$

What does this imply about the growth rates of capital and output?

The growth rate of capital is going to be equal to the growth rate of consumption since $(1-s)$ and A are constant over time. Also, since output is a linear function of capital and A is constant over time, output grows at the same rate as capital. So all three variables grow at the same rate.

c) Do k and c converge to a steady state? Is the economy on a balanced growth path?

This model does not have a steady state, but all variables grow at a constant rate. Thus, the economy is on a balanced growth path.

d) Find the optimal savings rate, s .

If we rewrite the resource constraint as follows

$$\frac{\dot{k}(t)}{k(t)} = A - \frac{c(t)}{k(t)} - \delta$$

and plug in our guess for consumption

$$\frac{\dot{k}(t)}{k(t)} = A - (1-s)A - \delta$$

and then use the fact that the growth rate of capital is equal to the growth rate of consumption, we get an equation that we can solve for s

$$\begin{aligned}\frac{1}{\theta}[A - \delta - \rho] &= sA - \delta \\ s &= \frac{A - \rho + (\theta - 1)\delta}{\theta A}\end{aligned}$$

Question 3. Knowledge Spillovers

This question is based on the Learning by Doing and Knowledge Spillovers model discussed in the class notes (Chapter 6.4) and Romer's textbook (page 120). The economy is described as follows

$$U = \int_{t=0}^{\infty} e^{-\rho t} \frac{c(t)^{1-\gamma}}{1-\gamma} dt$$

$$\dot{k}(t) = F(k(t), h(t)) - c(t) - \delta k(t)$$

$Y(t) = K(t)^\alpha (h(t)L(t))^{1-\alpha}$ with $0 < \alpha < 1$, where $h(t) = bk(t)$ with $b > 0$ and $L(t) = L$

a) Describe how human capital, as given by h , accumulates in this economy. Do firms or individuals directly invest in improving the level of human capital? Or, is it simply a side effect of physical capital accumulation? Given your answer, do you believe that the social planner and decentralized competitive equilibrium will coincide in this model?

The accumulation of human capital in this economy is simply a side effect of the accumulation of physical capital. The more physical capital per unit of labor, k , the more human capital per unit of labor. Thus, the more physical capital workers have, the more human capital they will accumulate by working with that capital. They learn more by working with more physical capital... i.e. learning by doing.

Since human capital is not the result of direct investment and is simply an externality from the creation of physical capital, we should not expect that the decentralized competitive equilibrium will coincide with the social planner problem. In fact, we should expect that an underinvestment in physical capital will occur in the decentralized competitive equilibrium.

b) Solve the social planner's problem in this economy. What is the growth rate of consumption?

First, we rewrite the production function in intensive form and plug in all the information we have about human capital

$$y(t) = k(t)^\alpha (h(t))^{1-\alpha} = k(t)^\alpha (bk(t))^{1-\alpha} = b^{1-\alpha} k(t)$$

Then, we set up the Hamiltonian, where we also plug the production function into the resource constraint

$$H = e^{-\rho t} \frac{c(t)^{1-\gamma}}{1-\gamma} + \mu(t)[b^{1-\alpha} k(t) - c(t) - \delta k(t)]$$

The FOCs are

$$\begin{aligned}H_c &= e^{-\rho t} c(t)^{-\gamma} - \mu(t) = 0 \\ H_k &= \mu(t)[b^{1-\alpha} - \delta] = -\dot{\mu}(t)\end{aligned}$$

As in Question 1, use the FOC for consumption to find $\mu(t)$ and $\dot{\mu}(t)$ and plug into the FOC for capital. This gives the following growth rate of consumption

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\gamma} [b^{1-\alpha} - \delta - \rho]$$

c) Solve for the competitive equilibrium in this economy. [For simplicity, assume that the labor supply of individuals is exogenous and equal to 1.]

We again set up the Hamiltonian, but now we need to use the household budget constraint instead of the resource constraint

$$H = e^{-\rho t} \frac{c(t)^{1-\gamma}}{1-\gamma} + \mu(t)[w(t) + r(t)k(t) - c(t) - \delta k(t)]$$

1, Use the FOCs of the household to find the Euler equation.

The FOCs are

$$H_c = e^{-\rho t} c(t)^{-\gamma} - \mu(t) = 0$$

$$H_k = \mu(t)[r(t) - \delta] = -\dot{\mu}(t)$$

And the resulting Euler equation is

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\gamma}[r(t) - \delta - \rho]$$

2. Using the firm's profit maximizing behavior, what is the equilibrium interest rate for physical capital and the equilibrium wage rate? (Remember, the firm takes human capital as exogenous).

Firms solve the following problem

$$\max_{L(t), K(t)} K(t)^\alpha (h(t)L(t))^{1-\alpha} - w(t)L(t) - r(t)K(t)$$

Note that firms are not aware of the spillover effect that physical capital has on human capital. This implies that firms don't use $h(t) = bk(t)$ in their maximization problem. We only plug in for h in the FOCs for the firm, to find the competitive equilibrium of the economy.

The FOCs are

$$w(t) = (1-\alpha)k(t)^\alpha h(t)^{-\alpha} = (1-\alpha)b^{1-\alpha}k(t)$$

$$r(t) = \alpha k(t)^{\alpha-1} h(t)^{1-\alpha} = \alpha b^{1-\alpha}$$

3. What is the growth rate of consumption?

Plugging in for the interest rate we get

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\gamma}[\alpha b^{1-\alpha} - \delta - \rho]$$

d) Compare the growth rate of the social planner's equilibrium from b) to that of the decentralized equilibrium in c). How are they different, and why?

Clearly, the growth rate in the decentralized competitive equilibrium is lower. The reason for this is that competitive firms do not take into account that investing in more capital will have an externality in increasing the human capital stock. The social planner, however, takes into account this externality from physical capital when it chooses the optimal amount of physical capital. Hence, the decentralized equilibrium has a lower growth rate because firms under-invest in physical capital relative to a social planner.

e) Suppose we introduce a government into the competitive equilibrium problem. What could the government do to ensure that the competitive equilibrium growth rate of consumption coincides with the social planner's outcome?

As you saw in Question 1, imposing a tax on capital changes the RHS of the Euler equation. In particular, the government could impose a tax on capital, such that $1-\tau^k = \frac{1}{\alpha}$ which would make the competitive equilibrium growth rate of consumption equal to the social planner's. If we solve for τ^k , we find $\tau^k = 1 - \frac{1}{\alpha}$ and since $0 < \alpha < 1$, this tax is actually negative, which means that it's a subsidy. The government could thus impose a tax on labor income, τ^l , (as long as labor supply is exogenous this won't affect the Euler equation), and could use the money from that tax to subsidize investment in physical capital.