14.06 Problem Set 6
Spring 2005

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Question 1

Exercise 10.12 from Romer's textbook.
Answer.

(a) The policy maker wants to choose inflation in order to maximize her objective function, which is given by $W = c\gamma y - -(a\pi^2/2)$, subject to output being given by the supply function $y = \overline{y} + b(\pi - \pi^e)$. Thus the policymaker's problem is

$$\max_{\pi} \quad W = c\gamma\left[\overline{y} + b(\pi - \pi^e)\right] - \left(a\pi^2/2\right)$$

The FOC is

$$\frac{\partial W}{\partial \pi} = bc\gamma - a\pi = 0. \tag{1}$$

Thus the policymaker's choice of π is

$$\pi = \frac{bc\gamma}{a}. \tag{2}$$

(b) The public knows the policymaker will set inflation accordingly to equation(2). Thus with rational expectations, expected inflation must equal the expectation of the right-hand side of equation (2):

$$\pi^e = \mathrm{E}\left[\frac{bc\gamma}{a}\right] = \frac{bc\mathrm{E}[\gamma]}{a} = \frac{bc\overline{\gamma}}{a} \tag{3}$$

(c) The true social welfare function is given by $W^{SOC} = \gamma y - (a\pi^2/2)$. Taking he expectation of both sides of this expression with respect to the public's information set, so that γ is random, gives us

$$\mathrm{E}\left[W^{SOC}\right] = \mathrm{E}\left[\gamma\left(\overline{y} + b(\pi - \pi^e)\right) - \left(a\pi^2/2\right)\right], \tag{4}$$

where we have substituted for $y = \overline{y} + b(\pi - \pi^e)$. Now we substitute the policymaker's choice of π, equation (2), and the public's expectation of inflation, equation (3), into equation (4):

$$\mathrm{E}\left[W^{SOC}\right]=\mathrm{E}\left[\gamma\left[\overline{y}+b\left(\frac{bc\gamma}{a}-\frac{bc\overline{\gamma}}{a}\right)\right]-\frac{ab^2c^2\gamma^2}{2a^2}\right]. \tag{5}$$

Simplifying yields and using the fact that $\mathrm{E}[\gamma]=\overline{\gamma}$ we obtain

$$\mathrm{E}\left[W^{SOC}\right]=\overline{y\gamma}+\frac{b^2c}{a}\left[\mathrm{E}\left[\gamma^2\right]-\overline{\gamma}^2\right]-\frac{b^2c^2\mathrm{E}\left[\gamma^2\right]}{2a}. \tag{6}$$

Now define $\sigma_\gamma^2=\mathrm{Var}[\gamma]$ and use the fact that $\mathrm{E}[\gamma^2]=\mathrm{Var}[\gamma]+(\mathrm{E}[\gamma])^2$ to rewrite equation (6) as follows:

$$\mathrm{E}\left[W^{SOC}\right]=\overline{y\gamma}+\frac{b^2c}{a}\sigma_\gamma^2-\frac{b^2c^2}{2a}\left(\sigma_\gamma^2+\overline{\gamma}^2\right). \tag{7}$$

(d) To find the FOCs for the maximization, use equation (7) to set the derivative of the expected value of the social welfare function with respect to c equal to 0:

$$\frac{\partial\mathrm{E}\left[W^{SOC}\right]}{\partial c}=\frac{b^2}{a}\sigma_\gamma^2-\frac{b^2c}{a}\left(\sigma_\gamma^2+\overline{\gamma}^2\right)=0. \tag{8}$$

Solving for c yields

$$c=\frac{\sigma_\gamma^2}{\sigma_\gamma^2+\overline{\gamma}}=\frac{\sigma_\gamma^2}{\mathrm{E}\left[\gamma^2\right]}. \tag{9}$$

There is a trade-off here. From equation (2), we can see that choosing a more "conservative" policymaker, that is, one with a low c, produces a better performance in terms of average inflation. However, such a policymaker would not respond well to shocks. Thus there is some optimal level of "conservatism" that balances these two forces.

The value of c that maximizes the expected value of true social welfare is decreasing in the mean of γ. Since we know that π^e will equal π on average (since γ will equal $\overline{\gamma}$ on average), output will equal full employment output on average, regardless of the values of c or $\overline{\gamma}$. From equation (2), we can see that if γ is higher on average, inflation will also be higher on average, for a given c. thus it will be welfare-improving to offset this and keep inflation lower on average by having a policymaker with a lower c; that is, having a more "conservative" policymaker.

However, the value of c that maximizes expected social welfare is increasing in the variance of the γ shock. The more variable the shock is, the less "conservative" the central bank should be. Since the policymaker can act after γ is realized, she can choose to offset any deviation in γ from its expected value, which will raise welfare. The policymaker will do this only to the extent that she cares about the shock's effect. Thus the more that γ varies, the better it is to have a policymaker who cares about the shock's effect and will act to offset it. ∎

Question 2

Assume there are a large number of isolated farmers.[1] Each knows the size of his own crop, y_i. The size of the crop on any farm at any date is described by

$$y_i = \alpha + \varepsilon_i \tag{10}$$

where $Cov(\varepsilon_i, \varepsilon_j) = 0 \; \forall i \neq j$, α and ε_i are independent, $\alpha \sim \text{Normal}\left(\overline{\alpha}, \sigma_\alpha^2\right)$, and $\varepsilon_i \sim Normal\left(0, \sigma_\varepsilon^2\right)$. Thus if $Y \equiv \sum_{i=1}^n y_i$, then $\text{E}\left(Y|y_i\right) = h_1 + nh_2 y_i$. Assume that there is a linear demand curve for the crop, so

$$Y = a - bP_s, \tag{11}$$

where P_s is the spot price next period.

(a) Show that the distribution of P_s for a given y_i is normal with mean $\left(a - \text{E}\left[Y|y_i\right]\right)/b$ and variance σ_p^2.

Answer. Rewrite equation (11):

$$P_s = \frac{a - Y}{b}. \tag{12}$$

Now use the facts that, conditional on y_i, Y is normally distributed, and that any linear transformation will also be normally distributed. In this particular case we have that conditional on y_i, $P_s \sim \text{Normal}\left(\frac{a - \mathbb{E}(Y|y_i)}{b}, \sigma_p^2\right)$, where σ_p^2 is the appropriate transformation of the variance of Y conditional on y_i. ▪

Since the individuals differ in their expectations about the spot prices in the next period, there is an incentive to set up a futures market. In this market the farmers trade today rights on the crops tomorrow at a price P_f.

(b) The demand for "futures" by farmer i is given by

$$Y_i^f = \left[\frac{\frac{a - \text{E}(Y|y_i)}{b} - P_f}{k\sigma_p^2}\right] + y_i, \tag{13}$$

where k is a preferences parameter equal to all farmers.[2]

Find an expression for the aggregate demand for futures $\sum_{i=1}^n Y_i^f$. What is the market equilibrium condition?

Answer.

The aggregate demand for futures is just the sum of all the individual demands:

[1] Denote the number of farmers by n, and assume that it is sufficiently large so you can apply the Law of Large Numbers in case you need it.

[2] This is the exact demand for agents with constant absolute risk aversion preferences with risk aversion given by k.

$$
\begin{aligned}
Y^f &= \sum_{i=1}^{n} Y_i^f && (14)\\
&= \frac{n}{k\sigma_p^2}\left[\frac{a-h_1-h_2Y}{b} - P_f\right] + Y. && (15)
\end{aligned}
$$

The market equilibrium condition is $Y^f = 0$. The total endowment of "futures" is 0, so the net holdings of "futures" must be 0. It is true that agents expect to have a crop different from 0, but the assets being traded are rights on prices or quantities, not actual quantities. It is the same as the market clearing condition in the bonds market in the final exam. ∎

(c) Show that the equilibrium price P_f is a linear function

$$
P_f = h_3 + h_4 P_s. \tag{16}
$$

What can you say of P_f as a signal?
Hint: use equation (11).
Answer. Take the market equilibrium condition

$$
\begin{aligned}
0 &= \frac{n}{k\sigma_p^2}\left[\frac{a-h_1-h_2Y}{b} - P_f\right] + Y\\
Y &= \frac{n}{k\sigma_p^2}\left[P_f - \frac{a-h_1-h_2Y}{b}\right],
\end{aligned}
$$

and use equation (11) to write

$$
\begin{aligned}
a - bP_s &= \frac{n}{k\sigma_p^2}\left[P_f - h_2P_s - \frac{a-h_1-h_2a}{b}\right]\\
\left[h_2\frac{n}{k\sigma_p^2} - b\right]P_s &= \left(\frac{a-h_1-h_2a}{b}\right)\frac{n}{k\sigma_p^2} - a + \frac{n}{k\sigma_p^2}P_f
\end{aligned}
$$

rearranging terms we have

$$
P_f = h_3 + h_4 P_s. \tag{17}
$$

From equation (17) we can see that the equilibrium price of the futures is a perfect signal of the price in the spot market tomorrow, meaning that a single agent can reduce the uncertainty about the spot price tomorrow to 0 just by *observing* the futures *today*. ∎

(d) The result you just obtained is different from the one Angeletos and Werning obtain in their model with endogenous information. What is the main fundamental difference between the models that leads to the different results?
How would you modify this model in order to recover the same qualitative result

Angeletos and Werning obtain about prices as aggregators of private information? Explain which formula you would modify and provide intuition.

Answer. There is one main difference: in this case there is no noise in the aggregation process. The Law of Large Numbers cancels all the noise coming from the private signals, and there is no aggregate noise in the futures market. Thus, the equilibrium price for futures (the risky asset in Angeletos and Werning's model) delivers a perfect signal for the equilibrium in the spot market tomorrow.

Any modification has to lead us to obtain a noisy aggregate signal. There are various options, here goes one, change the market equilibrium condition saying that the net demand for the futures does not need to be 0, but it is equal to a random variable with mean 0, not observed by the agents (much as in Angeletos and Werning).

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(e) *Optional*

Assume that the timing is as follows. In the first period the farmers trade in the futures market and observe the price P_f, but they have the information about their own crop y_i. In the second period there is an spot market where the crops are sold and the price P_s is observed.

Suppose you are a farmer in this economy, if you can observe the price P_f at no cost, do you have any incentive to trade "futures" in the first period?

What does this tell you about the "futures market"?

Answer. As P_f gives you perfect info about the spot price tomorrow, there is no incentive to trade futures (which are assets that allow you to protect against uncertainty), therefore it is an open question who trades. If the market collapses there is no signal.

Originally, Grossman and Stiglitz (1976) mentioned this effect as one of the many reasons for market breakdowns to happen. If you are interested, read their original paper. ▪

References

Grossman, S., and J. Stiglitz (1976); "Information and Competitive Price Systems", *American Economic Review*, 66(2), May, p. 246-253.