14.06 Problem Set #5 Solutions

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1. **Consumption Smoothing with Certainty**

Consider the following utility function and lifetime budget constraint:

$$u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}, \quad \gamma > 0$$

$$\sum_{t=0}^{T} q_t c_t = q_0 (1+R_0) a_0 + \sum_{t=0}^{T} q_t y_t$$

$$q_t = \frac{q_{t-1}}{1+R_t}$$

(a) Assume that the agent's discount factor β is such that $1+R_t = 1/\beta$ for all t. (worth 2 points)

i) Prove that he optimal consumption path is constant

The lagrangian of the problem can be written as:

$$L = \sum_{t=0}^{T} \beta^t u(c_t) + \lambda \left[q_0 (1+R_0) a_0 + \sum_{t=0}^{T} q_t y_t - \sum_{t=0}^{T} q_t c_t \right]$$

The FOC for c_t is given by the following:

$$\beta^t u'(c_t) = \lambda q_t \quad \forall\, t$$

Combining two of the FOCs, we have the following relationship:

$$\frac{\beta^t u'(c_t)}{\beta^{t+1} u'(c_{t+1})} = \frac{\lambda q_t}{\lambda q_{t+1}}$$

This reduces to the following:

$$\frac{u'(c_t)}{u'(c_{t+1})} = \beta \frac{q_t}{q_{t+1}} = \beta(1+R) = 1$$

$$u'(c_t) = u'(c_{t+1})$$

$$c_t = c_{t+1} \quad \forall\, t$$

ii) Solve for the optimal consumption level in each period of life

Given that consumption is constant, we can plug $c_t = \bar{c}$ back into our budget constraint to solve for the optimal level of consumption.

$$\sum_{t=0}^{T} q_t \bar{c} = q_0 (1 + R_0) a_0 + \sum_{t=0}^{T} q_t y_t$$

$$\bar{c} = \frac{q_0 (1 + R_0) a_0 + \sum_{t=0}^{T} q_t y_t}{\sum_{t=0}^{T} q_t}$$

iii) Why was it necessary for us to assume $1 + R_t = 1/\beta$ for all t?

It was necessary to make this assumption so that individual's discount rate exactly matches the economy's interest rate. When this is true, then the optimal consumption path of the economy will be flat. If this wasn't true, then the optimal consumption path might be growing or decreasing with time.

To simplify the math, assume $R = 0$ and $q_0 = 1$ for the remainder of this question.

(b) Suppose the individual's income is constant in each period of life such that $y = \bar{y}$ $\forall t$. (worth 2 points)

i) What is the optimal consumption level in each period of life now?

From the above assumptions, we know that it must be the case that $q_t = 1$ $\forall t$. The optimal consumption level is now given by:

$$\bar{c} = \frac{a_0 + \sum_{t=0}^{T} y_t}{T+1} = \frac{a_0 + (T+1)\bar{y}}{T+1} = \frac{a_0}{T+1} + \bar{y}$$

ii) How much does the individual save in period t ?

By definition, $s_t = y_t - c_t$. Solving this, we have the following:

$$s_t = \bar{y} - \frac{a_0}{T+1} + \bar{y}$$

$$s_t = -\frac{a_0}{T+1}$$

iii) Are the savings positive or negative? Explain your answer.

When $a_0 > 0$, the savings are negative in this model. i.e. The individual actually consumes some of his initial assets each period. In fact, the individual lives a total of $T+1$ periods in this model, and he will find it optimal to consume a fraction $1/(T+1)$ of his initial assets in each period of life. The intuition behind this is straight forward: The individual wishes to smooth his consumption path perfectly. Since his income is constant each period of life, all he needs to do to smooth his consumption is to consume a constant fraction of his initial savings in each period of life since the interest rate is zero. When $a_0 < 0$, however, individuals begin life with debt, and saving is positive in each period as they slowly pay off the debt each period.

(c) **Now suppose that the government decides to tax individuals. The government implements a tax on income in each period of life such that an individual's income is now given by $(1-\tau)\bar{y}$, where $\tau = 1/(1+T)$. The government takes the revenue from the tax and throws it in the ocean.** (worth 2 points)

i) **How much did the tax reduce the individual's per-period (contemporaneous) income?**

The individual's per period income is reduced by $\tau\bar{y} = \frac{\bar{y}}{1+T}$.

ii) **How much did the tax reduce the individual's lifetime income?**

The individual's lifetime income was given by $\sum_{t=0}^{T} y_t = (T+1)\bar{y}$

The individual's lifetime income is now given by $(T+1)(1-\tau)\bar{y}$

Thus, lifetime income was reduced by $(T+1)\tau\bar{y} = \bar{y}$

iii) **How much does the optimal consumption at time $t = 0$ fall?**

The optimal consumption level after the tax is given by $\bar{c} = \frac{a_0}{T+1} + (1-\tau)\bar{y}$. Thus, consumption falls by $\tau\bar{y} = \frac{\bar{y}}{1+T}$ in each period of life.

(d) **Now suppose that the government taxes individual's income in only the last period of life when $t = T$. Specifically, assume that $\tau = 1$, in the last period of life, and $\tau = 0$ in all earlier periods.** (worth 2 points)

i) **How much did the tax reduce the individual's per-period (contemporaneous) income in each period?**

The per-period income falls by $\bar{y}$ only in the last period of life.

ii) **How much did the tax reduce the individual's lifetime income?**

Lifetime income is reduced by $\bar{y}$ just as in part (c).

iii) How much does the optimal consumption at time $t = 0$ fall?

The optimal consumption level after the tax is given by $\bar{c} = \frac{a_0}{T+1} + \bar{y} - \frac{\bar{y}}{1+T}$.

Thus, consumption falls by $\frac{\bar{y}}{1+T}$ in each period of life just as before.

(e) In a few sentences, explain the intuition of why the government taxes in parts (c) and (d) have the same affect on consumption at time $t = 0$. Be sure to discuss how changes in contemporaneous income and lifetime (permanent) income each affect consumption. (worth 2 points)

Individuals' consumption only responds to changes in permanent income. (This is the Permanent Income Hypothesis result we saw in class, where the marginal propensity to consume only depends on permanent income changes and not transitory changes.) In both tax regimes, the government tax has the same affect in permanent income, and therefore, it will have the same effect on consumption in each period of life. When the tax is actually taken from the individuals doesn't matter. i.e. Contemporaneous (or transitory) changes in income don't matter for consumption, only changes in permanent income matter for consumption!

2. Consumption Smoothing with Uncertainty

Consider the following utility function and lifetime budget constraint.

$$u\left(c\left(S^t\right)\right) = c\left(S^t\right) - b\frac{c\left(S^t\right)^2}{2}, \quad \text{b} > 0$$

$$\sum_{t=0}^{\infty}\sum_{S^t} q\left(S^t\right)c\left(S^t\right) \le \sum_{t=0}^{\infty}\sum_{S^t} q\left(S^t\right)y\left(S^t\right)$$

In this model of uncertainty, S^t represents the 'event' S, at time t, and all prices and consumption levels will be a function of S^t. There are two possible states of the world, $S \in$ {war, peace}. Individuals have a discount factor of β. For this question, you should assume that the individual's wealth and income stream are such that his or her consumption is always in the range where marginal utility is positive.

(a) Please describe (in a few sentences) what a 'good' is in this model of uncertainty. (i.e. what will the individual in this economy be optimizing over?) How does this differ from the first question of this problem set, which dealt with a model of certainty? How do the prices, $q(\ )$, in this economy differ? (worth 2 points)

In the first model of certainty, a 'good' per se was consumption at each point in time. i.e. Consumption at time t was a 'good', and consumption at time $t+1$ was a separate good. In the model of uncertainty, however, a good is consumption at a particular time *given* a certain state. i.e. Consumption at time $t=3$ given the 'state' of the world has been "peace" for time periods $t=0,1,2,3$, is one 'good'. Consumption at time $t=3$, given the 'state' of the world has been "peace" for time periods $t=0,1,2$, but "war" for $t=3$ is another good. Therefore, the consumption goods vary by time and state, whereas, in a model of certainty, they only vary by time. The prices in this model of uncertainty, $q(S^t)$, reflect this in that they are dependent on the state and time also.

(b) Interpret the lifetime budget constraint I have given you: (worth 2 points)

i) What was assumed about the initial assets an individual begins with?

I've implicitly assumed that the individual's begin with zero assets at the beginning of their life.

ii) Can an individual's consumption in any one state and time exceed his or her income in that state and time? If so, how is this possible?

Yes, it is entirely possible that $C(S^t) > Y(S^t)$ with out violating the budget constraint. Individuals can consume more than their income in a given state of the world, but in order to still satisfy their budget constraint, it must be that they consume less than their income in some other state of the world.

Note: This result doesn't really come from there not being a borrowing constraint as was the case in the model of certainty. Rather, the ability to transfer income across states is possible by our assumption that there is a sufficiently rich market of financial assets that pay off differently in each state. Or, you could think of this as a perfect insurance market.

(c) Solve for the FOCs of the individual's maximization problem. Remember to account for the probability, $\pi(S^t)$, of each event occurring when you write out the individual's total *expected* lifetime utility. (worth 2 points)

The langrangian can be written as:

$$L = \sum_{t=0}^{\infty}\sum_{S^t}\beta^t\pi(S^t)\left(c(S^t) - b\frac{c(S^t)^2}{2}\right) + \lambda\left[\sum_{t=0}^{\infty}\sum_{S^t}q(S^t)y(S^t) - \sum_{t=0}^{\infty}\sum_{S^t}q(S^t)c(S^t)\right]$$

Since individuals optimize (or choose) their consumption for each possible state of the world at each point in time, we get our first order conditions by taking the derivative with respect to $c(S^t)$ for all S^t and t and setting it equal to zero. This gives us the following:

$\beta^t \pi(S^t) u'(c(S^t)) = \lambda q(S^t) \quad \forall\, S^t$ and t

Rewritten, it becomes: $\beta^t \pi(S^t)(1 - bC(S^t)) = \lambda q(S^t) \quad \forall\, S^t$ and t

(d) Now consider two possible states of the world at time t. In the first state, which we'll call W^t, the world has been at war for all periods of time up until time t. In the second state, which we'll call P^t, the world has been at peace for all periods of time up until time t. Use your FOCs from earlier to prove the following condition is true. Explain the intuition behind this condition. You will not receive credit for this question if you don't explain the intuition! (worth 2 points)

$$\frac{\pi(P^t)(1 - bC(P^t))}{\pi(W^t)(1 - bC(W^t))} = \frac{q(P^t)}{q(W^t)}$$

This condition follows directly from our FOC in part (c). The intuition is that individuals equate the relative *expected* marginal utilities of consumption in the two states of the world (the LHS of the equation), with the relative prices in the two states of the world (the RHS). i.e. You equate the expected marginal rate of substitution with the marginal rate of transformation. Individuals equalize their 'bang for buck' in each period.

(e) If we assume that the interest rate is constant and equal to the individuals discount rate, it can be shown that $q(S^t) = \pi(S^t)\beta^t$ for all S and t. Assume this is true. Using your first order conditions from part (c), what can you say about how consumption varies across different states of the world? How is consumption smoothing different in this model of 'uncertainty' compared to consumption smoothing we find in the model of 'certainty'? (worth 2 points)

Using our FOC, $\beta^t \pi(S^t)(1 - bC(S^t)) = \lambda q(S^t) \quad \forall\, S^t$ and t, we now have the result that $C(S^t) = \frac{1 - \lambda}{b} \quad \forall\, S^t$ and t. Thus, we see that individuals will choose a consumption level that is equal across all events. From our earlier FOC, we know that individuals will wish to consume more in states that have a high probability of occurring. However, our assumption for this question makes the goods in these high probability states more expensive. This makes individuals less willing to consume in these states. In fact, the higher price will exactly offset the incentive to consume in the high probability state. Thus, once again, we have the result that individuals smooth their consumption, but whereas individuals only smoothed their consumption across time in the model of certainty, they now also smooth their consumption across the different possible events.