Lecture 5
Nash equilibrium & Applications

14.12 Game Theory

Muhamet Yildiz

Road Map

1. Rationalizability – summary
2. Nash Equilibrium
3. Cournot Competition
   1. Rationalizability in Cournot Duopoly
4. Bertrand Competition
5. Commons Problem
6. **Quiz**
7. Mixed-strategy Nash equilibrium

Dominant-strategy equilibrium

$s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$

Definition: s_i* **strictly dominates** s_i iff

$$u_i(s_i^*, s_{-i}) > u_i(s_i, s_{-i}) \quad \forall s_{-i};$$

s_i* **weakly dominates** s_i iff $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i}) \quad \forall s_{-i}$ and at least one of the inequalities is strict.

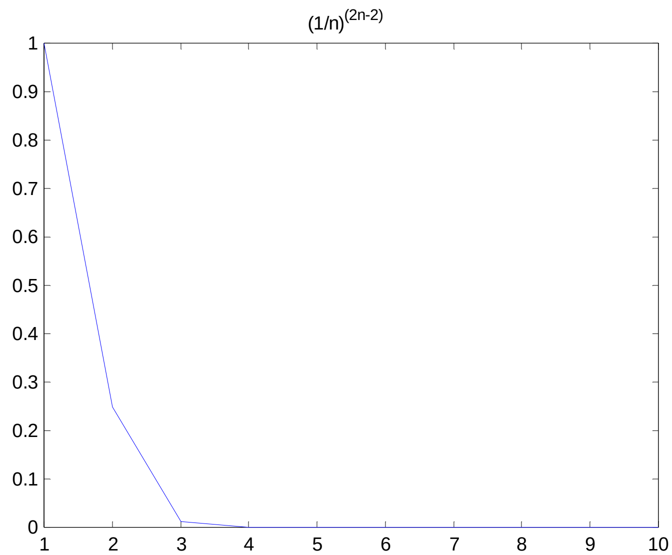
Definition: A strategy s_i* is a **dominant strategy** iff s_i* **weakly dominates** every other strategy s_i.

Definition: A strategy profile s* is a **dominant-strategy equilibrium** iff s_i* is a dominant strategy for each player i.

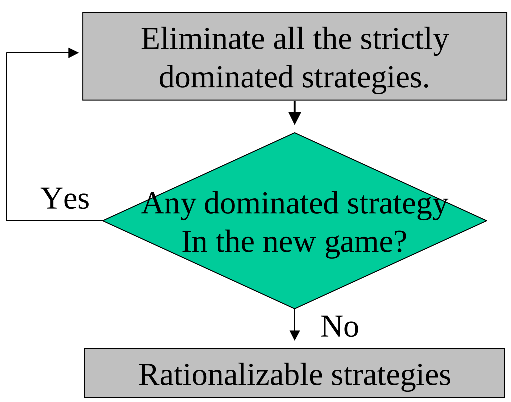
Examples: Prisoners' Dilemma; Second-Price auction.

Question

What is the probability that an nxn game has a dominant strategy equilibrium given that the payoffs are independently drawn from the same (continuous) distribution on [0,1]?



Rationalizability



The play is rationalizable, provided that …

Simplified price-competition

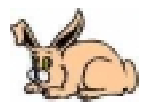
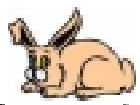
Firm 1 \ Firm 2	High	Medium	Low
High	6,6	0,10	0,8
Medium	10,0	5,5	0,8
Low	8,0	8,0	4,4

Dutta

A strategy profile is rationalizable when …

- Each player's strategy is consistent with his rationality, i.e., maximizes his payoff with respect to a conjecture about other players' strategies;
- These conjectures are consistent with the other players' rationality, i.e., if i conjectures that j will play s_j with positive probability, then s_j maximizes j's payoff with respect to a conjecture of j about other players' strategies;
- These conjectures are also consistent with the other players' rationality, i.e., …
- Ad infinitum

Stag Hunt

		
	(2,2)	(4,0)
	(0,4)	(6,6)

A summary

- If players are rational (and cautious), then they play the dominant-strategy equilibrium whenever it exists
  - But, typically, it does not exist
- If it is common knowledge that players are rational, then they will play a rationalizable strategy-profile
  - Typically, there are too many rationalizable strategies
- Now, a stronger assumption: The players are rational and their conjectures are mutually known.

Nash Equilibrium

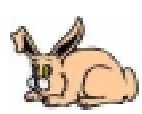
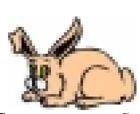
Definition: A strategy-profile s* =(s_1*,…,s_n*) is a **Nash Equilibrium** iff, for each player i, and for each strategy s_i, we have

$$u_i(s_1^*,\ldots,s_{i-1}^*,s_i^*,s_{i+1}^*,\ldots,s_n^*)$$
$$\geq u_i(s_1^*,\ldots,s_{i-1}^*,s_i\,,s_{i+1}^*,\ldots,s_n^*),$$

i.e., no player has any incentive to deviate if he knows what the others play.

??If players are rational, and their conjectures about what the others play are mutually known, then they must be playing a Nash equilibrium.

Stag Hunt

		
	(2,2)	(4,0)
	(0,4)	(6,6)

Economic Applications

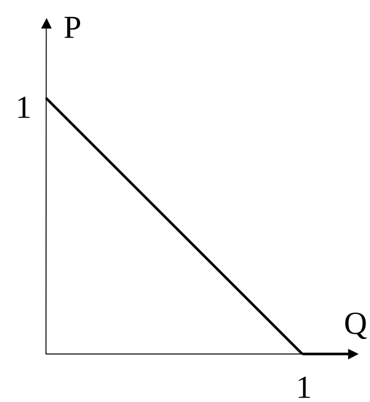
1. Cournot (quantity) Competition
   1. Nash Equilibrium in Cournot duopoly
   2. Nash Equilibrium in Cournot oligopoly
   3. Rationalizability in Cournot duopoly
2. Bertrand (price) Competition
3. Commons Problem

Cournot Oligopoly

- $N = \{1,2,\ldots,n\}$ firms;
- Simultaneously, each firm i produces q_i units of a good at marginal cost c,
- and sells the good at price

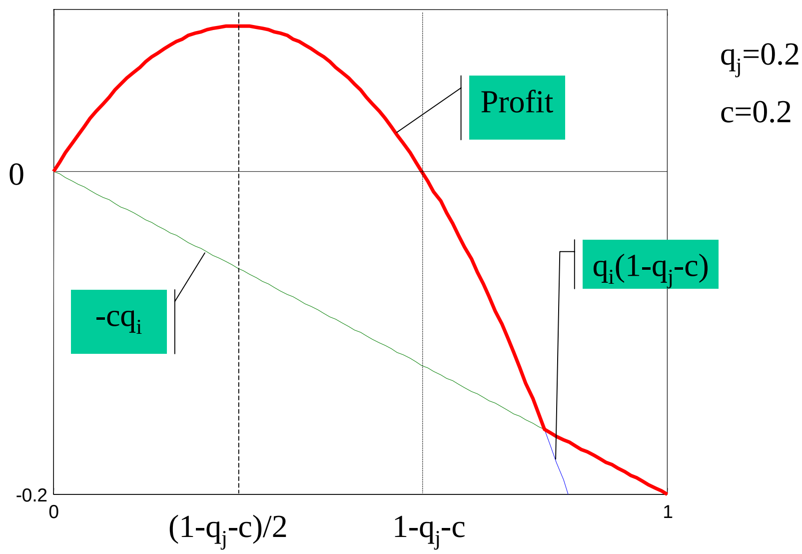
  $$P = \max\{0,1-Q\}$$

  where $Q = q_1+\ldots+q_n$.
- Game = $(S_1,\ldots,S_n; \pi_1,\ldots,\pi_n)$ where $S_i = [0,\infty)$,



$$\pi_i(q_1,\ldots,q_n) = \begin{cases} q_i[1-(q_1+\ldots+q_n)-c] & \text{if } q_1+\ldots+qn < 1, \\ -q_i c & \text{otherwise.} \end{cases}$$

Cournot Duopoly -- profit

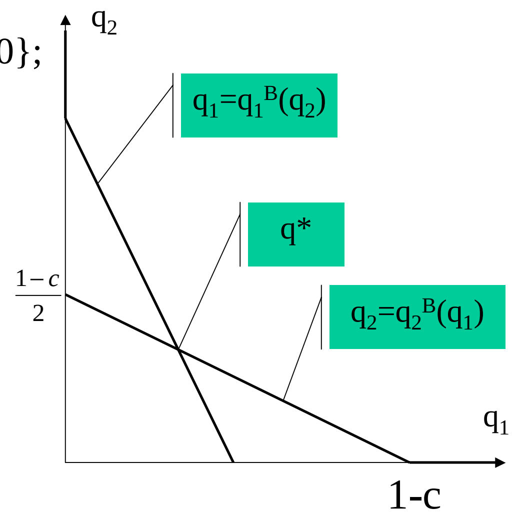


C-D – best responses

- $q_i^B(q_j) = \max\{(1-q_j-c)/2, 0\}$;
- Nash Equilibrium q*:

  $q_1^* = (1-q_2^*-c)/2$;

  $q_2^* = (1-q_1^*-c)/2$;
- $q_1^* = q_2^* = (1-c)/3$



Cournot Oligopoly --Equilibrium

- $q > 1-c$ is strictly dominated, so $q \le 1-c$.
- $\pi_i(q_1,\ldots,q_n) = q_i[1-(q_1+\ldots+q_n)-c]$ for each i.
- FOC:

$$\left.\frac{\partial \pi_i(q_1,\ldots,q_n)}{\partial q_i}\right|_{q=q^*} = \left.\frac{\partial [q_i(1-q_1-\cdots-q_n-c)]}{\partial q_i}\right|_{q=q^*}$$

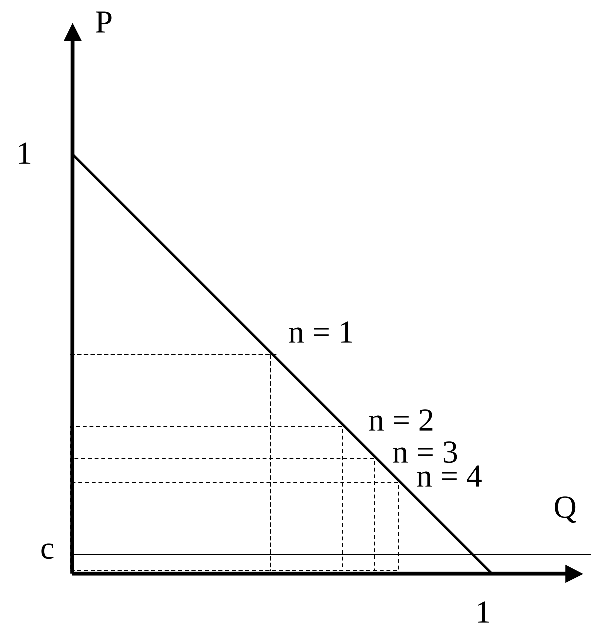
$$= (1-q_1^*-\cdots-q_n^*-c) - q_i^* = 0.$$

- That is,

$$\begin{aligned} 2q_1^* + q_2^* + \cdots + q_n^* &= 1-c \\ q_1^* + 2q_2^* + \cdots + q_n^* &= 1-c \\ &\vdots \\ q_1^* + q_2^* + \cdots + nq_n^* &= 1-c \end{aligned}$$

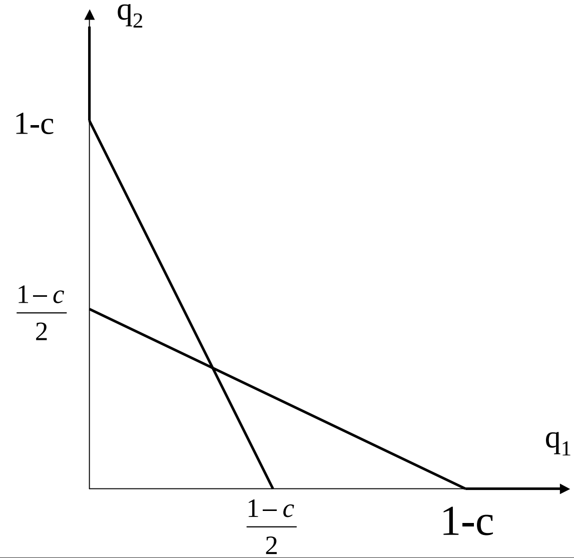
- Therefore, $q_1^* = \ldots = q_n^* = (1-c)/(n+1)$.

Cournot oligopoly – comparative statics



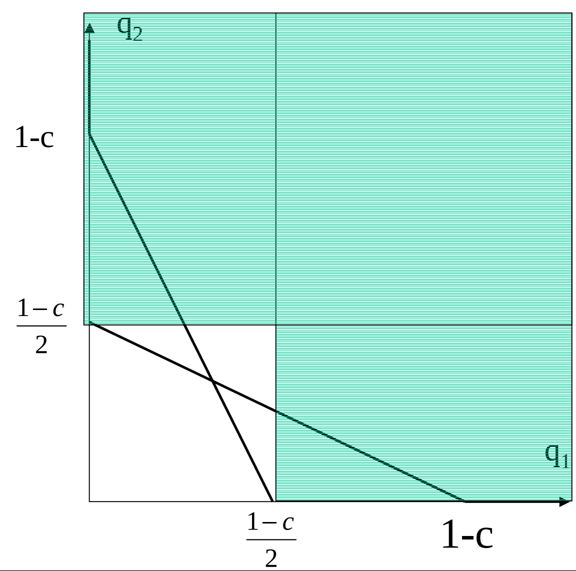
Rationalizability in Cournot Duopoly

Assume that players are rational.



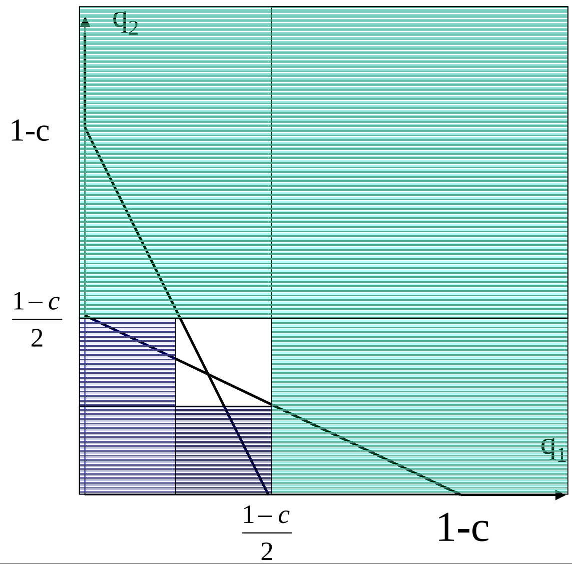
Players are rational:

Assume that players know this.



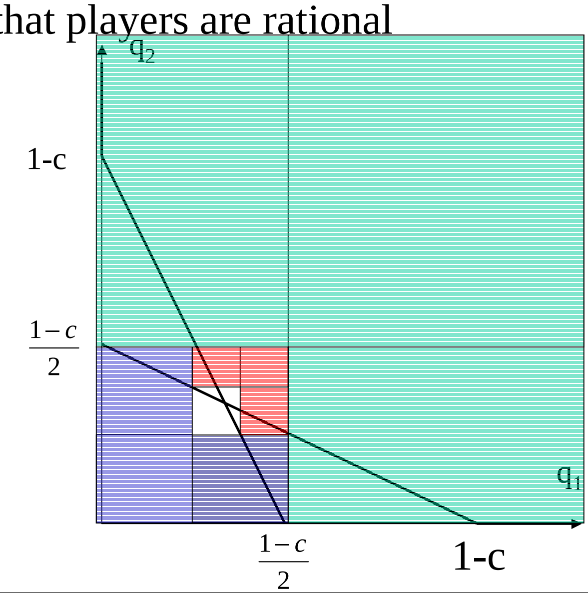
Players are rational and know that players are rational

Assume that players know this.



Players are rational; players know that players are rational; players know that players know that players are rational

Assume that players know this.



Rationalizability in Cournot duopoly

- If i knows that $q_j \le q$, then $q_i \ge (1\text{-}c\text{-}q)/2$.
- If i knows that $q_j \ge q$, then $q_i \le (1\text{-}c\text{-}q)/2$.
- We know that $q_j \ge q^0 = 0$.
- Then, $q_i \le q^1 = (1\text{-}c\text{-}q^0)/2 = (1\text{-}c)/2$ for each i;
- Then, $q_i \ge q^2 = (1\text{-}c\text{-}q^1)/2 = (1\text{-}c)(1\text{-}1/2)/2$ for each i;
- …
- Then, $q^n \le q_i \le q^{n+1}$ or $q^{n+1} \le q_i \le q^n$ where
  $q^{n+1} = (1\text{-}c\text{-}q^n)/2 = (1\text{-}c)(1\text{-}1/2+1/4\text{-}\ldots+(\text{-}1/2)^n)/2$.
- As $n\to\infty$, $q^n \to (1\text{-}c)/3$.

Bertrand (price) competition

- N = {1,2} firms.
- Simultaneously, each firm i sets a price p_i;
- If $p_i < p_j$, firm i sells $Q = \max\{1 - p_i,0\}$ unit at price p_i; the other firm gets 0.
- If $p_1 = p_2$, each firm sells Q/2 units at price p_1, where $Q = \max\{1 - p_1,0\}$.
- The marginal cost is 0.

$$\pi_1(p_1,p_2) = \begin{cases} p_1(1-p_1) & \text{if } p_1 < p_2 \\ p_1(1-p_1)/2 & \text{if } p_1 = p_2 \\ 0 & \text{otherwise.} \end{cases}$$

Bertrand duopoly -- Equilibrium

Theorem: The only Nash equilibrium in the “Bertrand game” is $p^* = (0,0)$.

Proof:

1. $p^*=(0,0)$ is an equilibrium.
2. If $p = (p_1,p_2)$ is an equilibrium, then $p = p^*$.
   1. If $p = (p_1,p_2)$ is an equilibrium, then $p_1 = p_2$...
      - If $p_i > p_j= 0$, for sufficiently small $\varepsilon>0$, $p_j' = \varepsilon$ is a better response to p_i for j. If $p_i > p_j> 0$, $p_i' = p_j$ is a better response for i.
   2. Given any equilibrium $p = (p_1,p_2)$ with $p_1 = p_2$, $p = p^*$.
      - If $p_1 = p_2>0$, for sufficiently small $\varepsilon>0$, $p_j' = p_j - \varepsilon$ is a better response to p_j for i.

Commons Problem

- $N = \{1,2,\ldots,n\}$ players, each with unlimited money;
- Simultaneously, each player i contributes $x_i \geq 0$ to produce $y = x_1+\ldots x_n$ unit of some public good, yielding payoff

$$U_i(x_i,y) = y^{1/2} - x_i.$$

Quiz

Each student i is to submit a real number x_i. We will pair the students randomly. For each pair (i,j), if $x_i \neq x_j$, the student who submits the number that is closer to $(x_i+x_j)/4$ gets 100; the other student gets 20. If $x_i = x_j$, then each of i and j gets 50.