Appendix A

Vector Algebra

As is natural, our Aerospace Structures will be described in a Euclidean three-dimensional space $\mathbb{R}^3$.

A.1 Vectors

A vector is used to represent quantities that have both *magnitude* and *direction*. Quantities that only need their magnitude to be fully represented are called scalars. We use bold letters to denote vectors, e.g. $\mathbf{v}$. The magnitude of a vector is denoted by $\|\mathbf{v}\|$.

A vector of unit length is called a unit vector. The unit vector in the direction of vector $\mathbf{v}$, is obtained by scaling the vector by the inverse of its magnitude:

$$\mathbf{e}_{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|},$$

We can also express $\mathbf{v}$ as

$$\mathbf{v} = \|\mathbf{v}\|\mathbf{e}_{\mathbf{v}}.$$

A.2 Components of a vector

A *basis* in $\mathbb{R}^3$ is a set of linearly independent vectors[1] such that any vector in the space can be represented as a linear combination of basis vectors. We will represent vectors in a *cartesian basis* where the basis vectors $\mathbf{e}_i$ are orthonormal, i.e. they have unit length and they are orthogonal with respect to each other. This can be expressed using dot products

$$\begin{array}{lll} \mathbf{e_1} \cdot \mathbf{e}_1 = 1 & \mathbf{e}_2 \cdot \mathbf{e}_2 = 1 & \mathbf{e}_2 \cdot \mathbf{e}_3 = 1 \\ \mathbf{e}_1 \cdot \mathbf{e}_2 = 0 & \mathbf{e}_1 \cdot \mathbf{e}_3 = 0 & \mathbf{e}_2 \cdot \mathbf{e}_3 = 0. \end{array}$$

[1] A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$ are *linearly dependent* if

$$\beta_1\mathbf{v}_1 + \beta_2\mathbf{v}_2 + \beta_3\mathbf{v}_3 + ... + \beta_n\mathbf{v}_n = 0$$

where β_1, β_2,...β_n are not all zero. In $\mathbb{R}^3$, the maximum number of linearly independent vectors is 3.

We can write these expressions in a very succinct form as follows

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

where the symbol δ_{ij} is the so-called

Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & : \text{ if } i = j \\ 0 & : \text{ if } i \neq j \end{cases}$$

Then we can represent any vector $\mathbf{v}$ in three dimensional space as follows

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = \sum_{i=1}^{3} v_i\mathbf{e}_i, \tag{A.1}$$

where v_1, v_2 and v_3 are the components of the vector in the basis $\mathbf{e}_i, i = 1, 3$

A.3 Indicial notation

Free index: A subscript index $()_i$ will be denoted a *free index* if it is not repeated in the same additive term where the index appears. *Free* means that the index represents **all** the values in its range.

- Latin indices will range from 1 to 3, $(i, j, k, ... = 1, 2, 3)$,

- Greek indices will range from 1 to 2, $(\alpha, \beta, \gamma, ... = 1, 2)$.

Examples:

1. $\mathbf{e}_i, i = 1, 2, 3$ can now be simply written as $\mathbf{e}_i$, no need to make the explicit mention of $i = 1, 2, 3,$ as i is a free index.
2. a_{i1} implies a_{11}, a_{21}, a_{31}. (one free index)
3. $x_\alpha y_\beta$ implies $x_1y_1, x_1y_2, x_2y_1, x_2y_2$ (two free indices).
4. a_{ij} implies $a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23}, a_{31}, a_{32}, a_{33}$ (two free indices implies 9 values).
5. $$\frac{\partial \sigma_{ij}}{\partial x_j} + b_i = 0$$

   has a free index (i), therefore it represents three equations:

   $$\frac{\partial \sigma_{1j}}{\partial x_j} + b_1 = 0$$
   $$\frac{\partial \sigma_{2j}}{\partial x_j} + b_2 = 0$$
   $$\frac{\partial \sigma_{3j}}{\partial x_j} + b_3 = 0$$

A.4 Summation Convention

In expressions such as:

$$\sum_{i=1}^{3} v_i \mathbf{e}_i,$$

we observe that the summation sign with its limits can be eliminated altogether if we adopt the convention that the summation is implied by the repeated index i. Then, a vector representation in a cartesian basis, Equation (A.1), can be shortened to

$$\mathbf{v} = \sum_{i=1}^{3} v_i \mathbf{e}_i = v_i \mathbf{e}_i,$$

More formally:

Summation convention: When a *repeated index* is found in an expression (inside an additive term) the summation of the terms ranging all the possible values of the indices is implied

Examples:

1. $a_i b_i = \sum_{i=1}^{3} a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$
2. $a_{kk} = a_{11} + a_{22} + a_{33}$.
3. $t_i = \sigma_{ij} n_j$ implies the **three** equations (why?):

$$t_1 = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3$$
$$t_2 = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3$$
$$t_3 = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3$$

Other important rules about indicial notation:

1. An index cannot appear more than twice in a single additive term, it's either free or repeated only once.

   $a_i = b_{ij} c_j d_j$ is INCORRECT

2. In an equation the *lhs* and *rhs*, as well as all the terms on both sides must have the same free indices

   - $a_i b_k = c_{ij} d_{kj}$ free indices i, k, CORRECT
   - $a_i b_k = c_{ij} d_{kj} + e_i f_{jj} + g_k p_i q_r$ INCORRECT, second term is missing free index k and third term has extra free index r
   - When the summation convention applies, the index is dummy (irrelevant): $a_i b_i = a_k b_k$.

A.5 Operations

Scalar product between vectors is defined as

$$\mathbf{a} \cdot \mathbf{b} = (a_i \mathbf{e}_i) \cdot (b_j \mathbf{e}_j) = a_i b_j (\mathbf{e}_i \cdot \mathbf{e}_j) = a_i b_j \delta_{ij} = a_i b_i.$$

Cross product between two basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ is defined as

$$\mathbf{e}_i \times \mathbf{e}_j = \varepsilon_{ijk} \mathbf{e}_k,$$

where ε_{ijk} is called the *alternating symbol* (or *permutation symbol*) and defined as follows

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } i, j, k \text{ are in cyclic order and not repeated } (123, 231, 312), \\ -1, & \text{if } i, j, k \text{ are not in cyclic order and not repeated} (132, 213, 321), \\ 0, & \text{if any of } i, j, k \text{ are repeated.} \end{cases}$$

In general, the cross product of two vectors can be expressed as

$$\mathbf{a} \times \mathbf{b} = (a_i \mathbf{e}_i) \times (b_j \mathbf{e}_j) = a_i b_j (\mathbf{e}_i \times \mathbf{e}_j) = a_i b_j \varepsilon_{ijk} \mathbf{e}_k.$$

$\varepsilon - \delta$ *identity* relates the Kronecker delta and the permutation symbol as follows

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}. \tag{A.2}$$

Problems:

1. Verify the $\varepsilon - \delta$ *identity* by the definition of Kronecker delta and the permutation symbol.

2. Use the $\varepsilon - \delta$ *identity* to verify $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$.

Dyadic product (or *tensor product*) between two basis vectors $\mathbf{e}_i$ and $\mathbf{e}_j$ defines a basis second order tensor $\mathbf{e}_i \otimes \mathbf{e}_j$ or simply $\mathbf{e}_i\mathbf{e}_j$. In general, the dyadic product

$$\mathbf{a} \otimes \mathbf{b} = (a_i \mathbf{e}_i) \otimes (b_j \mathbf{e}_j) = a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j \ ,$$

results in a second order tensor which has component $a_i b_j$ on the basis $\mathbf{e}_i \otimes \mathbf{e}_j$. The following identities are properties of the dyadic product

$$\begin{aligned}
(\alpha\mathbf{a}) \otimes \mathbf{b} &= \mathbf{a} \otimes (\alpha\mathbf{b}) = \alpha(\mathbf{a} \otimes \mathbf{b}), \text{ for scalar } \alpha \ ,\\
(\mathbf{a} \otimes \mathbf{b}) \cdot \mathbf{c} &= (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \ ,\\
\mathbf{a} \cdot (\mathbf{b} \otimes \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \ .
\end{aligned}$$

A.6 Transformation of basis

Given two orthonormal bases $\mathbf{e}_i, \tilde{\mathbf{e}}_k$ and a vector $\mathbf{v}$ whose components in each of these bases are v_i and $\tilde{v}_k$, respectively, we seek to express the components in basis in terms of the components in the other basis. Since the vector is unique:

$$\mathbf{v} = \tilde{v}_m \tilde{\mathbf{e}}_m = v_n \mathbf{e}_n$$

Taking the scalar product with $\tilde{\mathbf{e}}_i$:

$$\mathbf{v} \cdot \tilde{\mathbf{e}}_i = \tilde{v}_m (\tilde{\mathbf{e}}_m \cdot \tilde{\mathbf{e}}_i) = v_n (\mathbf{e}_n \cdot \tilde{\mathbf{e}}_i)$$

But $\tilde{v}_m (\tilde{\mathbf{e}}_m \cdot \tilde{\mathbf{e}}_i) = \tilde{v}_m \delta_{mi} = \tilde{v}_i$ from which we obtain:

$$\boxed{\tilde{v}_i = \mathbf{v} \cdot \tilde{\mathbf{e}}_i = v_j (\mathbf{e}_j \cdot \tilde{\mathbf{e}}_i)}$$

Note that $\mathbf{e}_j \cdot \tilde{\mathbf{e}}_i$ are the *direction cosines* of the basis vectors of one basis on the other basis:

$$\mathbf{e}_j \cdot \tilde{\mathbf{e}}_i = \|\mathbf{e}_j\| \|\tilde{\mathbf{e}}_i\| \cos \widehat{\mathbf{e}_j \tilde{\mathbf{e}}_i} = \cos \widehat{\mathbf{e}_j \tilde{\mathbf{e}}_i}$$

A.7 Tensors

Tensors are defined as the quantities that are independent of the selection of basis while the components transform following a certain rule as the basis changes.

When the basis changes from $\{\mathbf{e}_i\}$ to $\{\tilde{\mathbf{e}}_i\}$, a scalar does not change (e.g. mass), the vector component transforms as $\tilde{v}_i = v_j(\mathbf{e}_j \cdot \tilde{\mathbf{e}}_i)$ (see Section A.6), and the stress component transforms as $\tilde{\sigma}_{ij} = \sigma_{kl}\big(\mathbf{e}_k \cdot \tilde{\mathbf{e}}_i\big)\big(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_j\big)$ (see Equation 1.11), which defines a second order tensor. Similarly, the vector is a first order tensor, and a scalar is a zeroth order tensor. The order of the tensor equals the number of free indices (see Section A.3) that the tensor has, and it can also be interpreted as the dimensionality of the array needed to represent the tensor, as detailed in the next table:

# indices	Tensor order	Array type	Denoted as	Rule of Trans.
0	Zeroth	Scalar	α	
1	First	Vector	$\mathbf{v} = v_i\mathbf{e}_i = \tilde{v}_j\tilde{\mathbf{e}}_j$	$\tilde{v}_i = v_j(\mathbf{e}_j \cdot \tilde{\mathbf{e}}_i)$
2	Second	Matrix	$\boldsymbol{\sigma} = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \tilde{\sigma}_{ij}\tilde{\mathbf{e}}_i \otimes \tilde{\mathbf{e}}_j$	$\tilde{\sigma}_{ij} = \sigma_{kl}\big(\mathbf{e}_k \cdot \tilde{\mathbf{e}}_i\big)\big(\mathbf{e}_l \cdot \tilde{\mathbf{e}}_j\big)$

Higher order tensor can be defined following the transformation rule. For instance, the fourth order tensor can be defined as

$$\mathcal{C} = C_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l$$

with

$$\tilde{C}_{ijkl} = C_{pqrs}(\mathbf{e}_p \cdot \tilde{\mathbf{e}}_i)(\mathbf{e}_q \cdot \tilde{\mathbf{e}}_j)(\mathbf{e}_r \cdot \tilde{\mathbf{e}}_k)(\mathbf{e}_s \cdot \tilde{\mathbf{e}}_l)$$

upon the change of basis.

A.8 Tensor operations

Tensors are able to operate on tensors to produce other tensors. The *scalar product*, *cross product* and *dyadic product* of first order tensor (vector) have already been introduced in Sec A.5. In this section, focus is given to the operations related with the second order tensor.

Dot product with vector:

$$\boldsymbol{\sigma} \cdot \mathbf{a} = (\sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) \cdot (a_k\mathbf{e}_k) = \sigma_{ij}\mathbf{e}_i(\mathbf{e}_j \cdot a_k\mathbf{e}_k) = \sigma_{ij}\mathbf{e}_i a_k \delta_{jk} = \sigma_{ij}a_j\mathbf{e}_i$$

$$\mathbf{a} \cdot \boldsymbol{\sigma} = (a_k\mathbf{e}_k) \cdot (\sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) = a_k(\mathbf{e}_k \cdot (\sigma_{ij}\mathbf{e}_i))\mathbf{e}_j = a_k(\sigma_{ij}\delta_{ki})\mathbf{e}_j = a_i\sigma_{ij}\mathbf{e}_j$$

Double-dot product:

$$\boldsymbol{\sigma} : \boldsymbol{\epsilon} = (\sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) : (\epsilon_{kl}\mathbf{e}_k \otimes \mathbf{e}_l) = \sigma_{ij}\epsilon_{kl}(\mathbf{e}_i \cdot \mathbf{e}_k)(\mathbf{e}_j \cdot \mathbf{e}_l) = \sigma_{ij}\epsilon_{kl}\delta_{ik}\delta_{jl} = \sigma_{ij}\epsilon_{ij}$$

Double-dot product with fourth order tensor:

$$\mathcal{C} : \boldsymbol{\epsilon} = C_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l : \epsilon_{pq}\mathbf{e}_p \otimes \mathbf{e}_q = C_{ijkl}\epsilon_{pq}\mathbf{e}_i \otimes \mathbf{e}_j(\mathbf{e}_k \cdot \mathbf{e}_p)(\mathbf{e}_l \cdot \mathbf{e}_q) = C_{ijkl}\epsilon_{kl}\mathbf{e}_i \otimes \mathbf{e}_j\ .$$

Appendix B

Vector Calculus

B.1 nabla operator(∇)

In a Cartesian system with orthonormal basis $\{\mathbf{e}_i\}$, the nabla operator ∇ is denoted by

$$\nabla \equiv \mathbf{e}_i \frac{\partial}{\partial x_1} + \mathbf{e}_2 \frac{\partial}{\partial x_2} + \mathbf{e}_3 \frac{\partial}{\partial x_3} \ .$$

The *gradient* of a scalar field ϕ is defined as

$$\text{grad } \phi = \nabla \phi = \mathbf{e}_i \frac{\partial \phi}{\partial x_i} \ .$$

Problems:

1. Verify that $\nabla\phi$ is perpendicular to the surface $\{\mathbf{x}|\phi(x_i) = \text{const}\}$, and it is associated with the maximum spatial rate of change of ϕ. (Hint: the spatial rate of change of ϕ along unit vector $\hat{\mathbf{l}}$ is given by $\frac{\partial \phi}{\partial x_i}(\mathbf{e}_i \cdot \hat{\mathbf{l}})$.)

In general, the gradient of a tensor generates a new tensor with a higher order. For instance, the gradient of a first order tensor $\mathbf{v}$ (vector) produces a second order tensor:

$$\nabla \mathbf{v} = \mathbf{e}_i \frac{\partial}{\partial x_i} (v_j \mathbf{e}_j) = \frac{\partial v_j}{\partial x_i} \mathbf{e}_i \mathbf{e}_j = \frac{\partial v_j}{\partial x_i} \mathbf{e}_i \otimes \mathbf{e}_j.$$

It is also common to decompose the gradient of vector as

$$\nabla \mathbf{v} = \frac{1}{2} \left(\frac{\partial v_j}{\partial x_i} + \frac{\partial v_i}{\partial x_j} \right) \mathbf{e}_i \otimes \mathbf{e}_j + \frac{1}{2} \left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j} \right) \mathbf{e}_i \otimes \mathbf{e}_j.$$

The *divergence* of a vector is defined as

$$\text{div } \mathbf{v} = \nabla \cdot \mathbf{v} = (\mathbf{e}_i \frac{\partial}{\partial x_i}) \cdot (v_j \mathbf{e}_j) = \frac{\partial v_i}{\partial x_i}$$

which results in a scalar.

In general, the divergence of a tensor generates a tensor with a lower order. The divergence of a tensor $\Phi = \phi_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ produces a vector, as shown in next equations:

$$\begin{aligned} \nabla \cdot \Phi &= (\mathbf{e}_i \frac{\partial}{\partial x_i}) \cdot (\phi_{mn}\mathbf{e}_m \otimes \mathbf{e}_n) \\ &= \frac{\partial \phi_{in}}{\partial x_i}\mathbf{e}_n. \end{aligned}$$

The *curl* of a vector is defined as

$$\text{curl } \mathbf{v} = \nabla \times \mathbf{v} = \varepsilon_{ijk}\frac{\partial v_j}{\partial x_i}\mathbf{e}_k \ .$$

The Laplacian operator ∇^2 (sometimes written as Δ) on a scalar field ϕ is defined as the divergence of the gradient vector $\nabla\phi$:

$$\nabla^2\phi = \nabla \cdot (\nabla\phi) = (\nabla \cdot \nabla)\,\phi = \frac{\partial^2\phi}{\partial x_i \partial x_i} \ .$$

B.2 Integral Relations

The integral relations are equations that relate the volume integral to the surface integral, or the surface integral to the line integral. Consider an arbitrary region in space of volume V which is surrounded by surface S with a unit outer normal $\hat{\mathbf{n}}$.
Gradient Theorem: for any scalar field ϕ,

$$\int_V \text{grad } \phi \ dV = \oint_S \hat{\mathbf{n}}\phi \ dS \ . \tag{B.1}$$

Written in component,

$$\int_V \frac{\partial \phi}{\partial x_i} \ dV = \oint_S \hat{n}_i \phi \ dS \ .$$

Divergence Theorem (or Gauss' Theorem): for a first or second order tensor $\mathbf{A}$,

$$\int_V \text{div } \mathbf{A} \ dV = \oint_S \hat{\mathbf{n}} \cdot \mathbf{A} \ dS \tag{B.2}$$

For vector $\mathbf{A} = A_i\mathbf{e}_i$, the divergence theorem gives rise to one identity

$$\int_V \frac{\partial A_i}{\partial x_i} \ dV = \oint_S \hat{n}_i A_i \ dS \ .$$

While for second order tensor $\mathbf{A} = A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, the divergence theorem gives rises to three identities:

$$\int_V \frac{\partial A_{ij}}{\partial x_i} \ dV = \oint_S \hat{n}_i A_{ij} \ dS \ .$$

B.3 nabla operator in Cylindrical and Spherical Coordinate Systems

The nabla operator ∇ has already been introduced in Cartesian coordinate system in the previous section. Let $\hat{\mathbf{e}}_x$ be a unit vector along the x-axis, and define $\hat{\mathbf{e}}_y$ and $\hat{\mathbf{e}}_z$ by analogy. Then $\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\}$ forms a basis of the space. On the basis, the nabla operator is denoted as

$$\nabla = \hat{\mathbf{e}}_x \frac{\partial}{\partial x} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z} \, .$$

The Laplacian operator is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

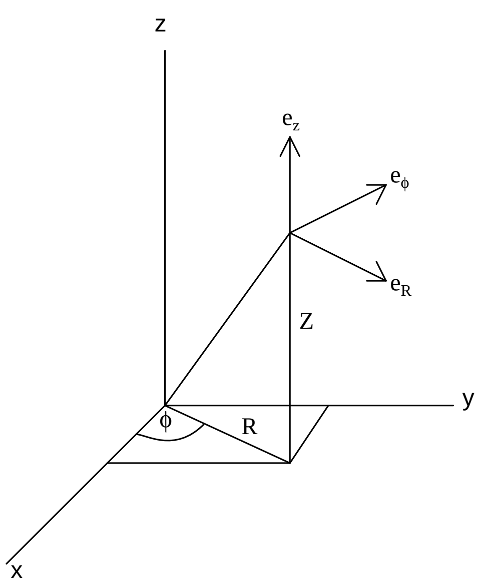
In cylindrical and spherical coordinate systems, the basis is defined locally, and will change as the position changes. As as a result, the nabla operator has different expressions.

B.3.1 Cylindrical coordinates

In a cylindrical coordinate system (R, ϕ, z), the orthonormal basis vectors associated with the coordinates are defined by

$$\begin{aligned} \hat{\mathbf{e}}_R &= \cos(\phi)\hat{\mathbf{e}}_x + \sin(\phi)\hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_\phi &= -\sin(\phi)\hat{\mathbf{e}}_x + \cos(\phi)\hat{\mathbf{e}}_y \\ \hat{\mathbf{e}}_z &= \hat{\mathbf{e}}_z \end{aligned}$$

where $\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\}$ are the three basis vectors in the Cartesian coordinate system (x, y, z).



The nabla operator in cylindrical coordinate system is denoted as:

$$\nabla = \hat{\mathbf{e}}_R \frac{\partial}{\partial R} + \hat{\mathbf{e}}_\phi \frac{1}{R} \frac{\partial}{\partial \phi} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z}$$

Factor $\frac{1}{R}$ in the second term is a little surprise at the first sight, while it makes the *rhs* dimensionally consistent. The new expression can be derived from the coordinate transformation as below.

(R, ϕ, z) coordinates can be written as functions of (x, y, z):

$$\begin{aligned}
R &= \sqrt{x^2+y^2} \\
\phi &= \arctan\frac{y}{x} \\
z &= z
\end{aligned}$$

which means given any scalar field $\psi(R,\phi,z)$, $\psi(R(x,y,z),\phi(x,y,z),z)$ is a scalar field in (x, y, z) coordinate system. Recalling the expression of ∇ in Cartesian coordinate system,

$$\begin{aligned}
\nabla\psi &= \hat{\mathbf{e}}_x\frac{\partial\psi}{\partial x} + \hat{\mathbf{e}}_y\frac{\partial\psi}{\partial y} + \hat{\mathbf{e}}_z\frac{\partial\psi}{\partial z} \\
&= \hat{\mathbf{e}}_x(\frac{\partial\psi}{\partial R}\frac{\partial R}{\partial x} + \frac{\partial\psi}{\partial\phi}\frac{\partial\phi}{\partial x}) + \hat{\mathbf{e}}_y(\frac{\partial\psi}{\partial R}\frac{\partial R}{\partial y} + \frac{\partial\psi}{\partial\phi}\frac{\partial\phi}{\partial y}) + \hat{\mathbf{e}}_z\frac{\partial\psi}{\partial z} \\
&= \frac{\partial\psi}{\partial R}\left(\hat{\mathbf{e}}_x\frac{\partial R}{\partial x} + \hat{\mathbf{e}}_y\frac{\partial R}{\partial y}\right) + \frac{\partial\psi}{\partial\phi}\left(\hat{\mathbf{e}}_x\frac{\partial\phi}{\partial x} + \hat{\mathbf{e}}_y\frac{\partial\phi}{\partial y}\right) + \hat{\mathbf{e}}_z\frac{\partial\psi}{\partial z} \\
&= \frac{\partial\psi}{\partial R}\left(\hat{\mathbf{e}}_x\frac{x}{R} + \hat{\mathbf{e}}_y\frac{y}{R}\right) + \frac{\partial\psi}{\partial\phi}\left(\hat{\mathbf{e}}_x(-\frac{y}{R^2}) + \hat{\mathbf{e}}_y\frac{x}{R^2}\right) + \hat{\mathbf{e}}_z\frac{\partial\psi}{\partial z} \\
&= \frac{\partial\psi}{\partial R}(\cos(\phi)\hat{\mathbf{e}}_x + \sin(\phi)\hat{\mathbf{e}}_y) + \frac{\partial\psi}{\partial\phi}\frac{1}{R}(-\sin(\phi)\hat{\mathbf{e}}_x + \cos(\phi)\hat{\mathbf{e}}_y) + \hat{\mathbf{e}}_z\frac{\partial\psi}{\partial z} \\
&= \hat{\mathbf{e}}_R\frac{\partial\psi}{\partial R} + \hat{\mathbf{e}}_\phi\frac{1}{R}\frac{\partial\psi}{\partial\phi} + \hat{\mathbf{e}}_z\frac{\partial\psi}{\partial z}\ ,
\end{aligned}$$

which denotes the ∇ operator in (R, ϕ, z) coordinates on the basis $\{\hat{\mathbf{e}}_R, \hat{\mathbf{e}}_\phi, \hat{\mathbf{e}}_z\}$.

Consequently, Laplacian operator on a scalar field ψ can be written as

$$\begin{aligned}
\nabla^2\psi &= \nabla\cdot(\nabla\psi) \\
&= (\hat{\mathbf{e}}_R\frac{\partial}{\partial R} + \hat{\mathbf{e}}_\phi\frac{1}{R}\frac{\partial}{\partial\phi} + \hat{\mathbf{e}}_z\frac{\partial}{\partial z})\cdot(\hat{\mathbf{e}}_R\frac{\partial\psi}{\partial R} + \hat{\mathbf{e}}_\phi\frac{1}{R}\frac{\partial\psi}{\partial\phi} + \hat{\mathbf{e}}_z\frac{\partial\psi}{\partial z}) \\
&= \frac{\partial^2\psi}{\partial R^2} + \hat{\mathbf{e}}_\phi\cdot\frac{1}{R}\frac{\partial}{\partial\phi}(\hat{\mathbf{e}}_R)\frac{\partial\psi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \\
&= \frac{\partial^2\psi}{\partial R^2} + \hat{\mathbf{e}}_\phi\cdot\frac{1}{R}\hat{\mathbf{e}}_\phi\frac{\partial\psi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \\
&= \frac{\partial^2\psi}{\partial R^2} + \frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \\
&= \frac{1}{R}\frac{\partial}{\partial R}(R\frac{\partial\psi}{\partial R}) + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}\ .
\end{aligned}$$

B.3.2 Strain and Stress components in cylindrical coordinates

The cylindrical coordinates have very important applications in plane stress and plane strain problems. Consider only the in-plane coordinates (*polar coordinates*) (r, θ). The displacement fields (u_r, u_θ) can be expressed as

$$\begin{aligned} u_r &= \cos\theta\, u_1 + \sin\theta\, u_2 \ , \\ u_\theta &= -\sin\theta\, u_1 + \cos\theta\, u_2 \end{aligned}$$

where (u_1, u_2) are the displacements in the Cartesian coordinates system. At the same time, it is easy to see that (u_1, u_2) can be expressed in terms of (u_r, u_θ) as

$$\begin{aligned} u_1(r,\theta) &= \cos\theta\, u_r - \sin\theta\, u_\theta \ , \\ u_2(r,\theta) &= \sin\theta\, u_r + \cos\theta\, u_\theta \ . \end{aligned}$$

The strain tensor $(\epsilon_{rr}, \epsilon_{r\theta}; \epsilon_{\theta r}, \epsilon_{\theta\theta})$ can be obtained through the rotation of the Cartesian coordinates by θ, i.e.

$$\begin{aligned} \epsilon_{rr} &= \epsilon_{11}\cos^2\theta + \epsilon_{22}\sin^2\theta + \epsilon_{12}\sin 2\theta \\ \epsilon_{\theta\theta} &= \epsilon_{11}\sin^2\theta + \epsilon_{22}\cos^2\theta - \epsilon_{12}\sin 2\theta \\ \epsilon_{r\theta} = \epsilon_{\theta r} &= -\tfrac{\epsilon_{11}-\epsilon_{22}}{2}\sin 2\theta + \epsilon_{12}\cos 2\theta \ . \end{aligned}$$

Using the definition and the chain rule derivative, the strains ϵ_{ij} can be expressed in terms of $(r, \theta, u_r, u_\theta)$, and finally the strains in polar coordinates can be obtained by substitution. From chain rule, it is straightforward to show that $\frac{\partial r}{\partial x_1} = \cos\theta$, $\frac{\partial r}{\partial x_2} = \sin\theta$, $\frac{\partial \theta}{\partial x_1} = -\frac{\sin\theta}{r}$, $\frac{\partial \theta}{\partial x_2} = \frac{\cos\theta}{r}$. And consequently,

$$\begin{aligned} \epsilon_{11} &= \frac{\partial u_1}{\partial x_1} = \frac{\partial u_1}{\partial r}\cos\theta + \frac{\partial u_1}{\partial \theta}(-\frac{\sin\theta}{r}) \ , \\ \epsilon_{22} &= \frac{\partial u_2}{\partial x_2} = \frac{\partial u_2}{\partial r}\sin\theta + \frac{\partial u_2}{\partial \theta}(\frac{\cos\theta}{r}) \ , \\ \epsilon_{12} &= \frac{1}{2}(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}) \\ &= \frac{1}{2}\left(\frac{\partial u_1}{\partial r}(\sin\theta) + \frac{\partial u_1}{\partial \theta}(\frac{\cos\theta}{r}) + \frac{\partial u_2}{\partial r}(\cos\theta) + \frac{\partial u_2}{\partial \theta}(-\frac{\sin\theta}{r})\right) \ . \end{aligned}$$

Since $\sin 2\theta = 2\sin\theta\cos\theta$,

$$\begin{aligned} \epsilon_{rr} =& \frac{\partial u_1}{\partial r}\cos^3\theta + \frac{\partial u_1}{\partial \theta}(-\frac{\sin\theta\cos^2\theta}{r}) + \frac{\partial u_2}{\partial r}\sin^3\theta + \frac{\partial u_2}{\partial \theta}(\frac{\cos\theta\sin^2\theta}{r}) \\ &+ \frac{\partial u_1}{\partial r}(\sin^2\theta\cos\theta) + \frac{\partial u_1}{\partial \theta}(\frac{\cos^2\theta\sin\theta}{r}) + \frac{\partial u_2}{\partial r}(\cos^2\theta\sin\theta) + \frac{\partial u_2}{\partial \theta}(-\frac{\sin^2\theta\cos\theta}{r}) \\ =& \frac{\partial u_1}{\partial r}\cos\theta + \frac{\partial u_2}{\partial r}\sin\theta \\ =& \frac{\partial(u_1\cos\theta + u_2\sin\theta)}{\partial r} \\ =& \frac{\partial u_r}{\partial r} \ . \end{aligned}$$

Similarly,

$$\begin{aligned}
\epsilon_{\theta\theta} =& \frac{\partial u_1}{\partial r}\cos\theta\sin^2\theta + \frac{\partial u_1}{\partial \theta}(-\frac{\sin^3\theta}{r}) + \frac{\partial u_2}{\partial r}\sin\theta\cos^2\theta + \frac{\partial u_2}{\partial \theta}(\frac{\cos^3\theta}{r}) \\
& - \frac{\partial u_1}{\partial r}(\sin^2\theta\cos\theta) - \frac{\partial u_1}{\partial \theta}(\frac{\cos^2\theta\sin\theta}{r}) - \frac{\partial u_2}{\partial r}(\cos^2\theta\sin\theta) + \frac{\partial u_2}{\partial \theta}(\frac{\sin^2\theta\cos\theta}{r}) \\
=& \frac{\partial u_1}{\partial \theta}(-\frac{\sin\theta}{r}) + \frac{\partial u_2}{\partial \theta}\frac{\cos\theta}{r} \\
=& \frac{1}{r}\left(\frac{\partial(-u_1\sin\theta + u_2\cos\theta)}{\partial\theta} + u_1\cos\theta + u_2\sin\theta\right) \\
=& \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) ,
\end{aligned}$$

and

$$\begin{aligned}
\epsilon_{\theta r} =& -\left(\frac{\partial u_1}{\partial r}\cos\theta - \frac{\partial u_1}{\partial \theta}\frac{\sin\theta}{r} - \frac{\partial u_2}{\partial r}\sin\theta - \frac{\partial u_2}{\partial \theta}\frac{\cos\theta}{r}\right)\sin\theta\cos\theta \\
& + \left(\frac{\partial u_1}{\partial r}(\sin\theta) + \frac{\partial u_1}{\partial \theta}(\frac{\cos\theta}{r}) + \frac{\partial u_2}{\partial r}(\cos\theta) + \frac{\partial u_2}{\partial \theta}(-\frac{\sin\theta}{r})\right)(\cos^2\theta - \frac{1}{2}) \\
=& \frac{1}{2}\left(\frac{\partial u_1}{\partial r}(-\sin\theta) + \frac{\partial u_1}{\partial \theta}\frac{\cos\theta}{r} + \frac{\partial u_2}{\partial r}\cos\theta + \frac{\partial u_2}{\partial \theta}\frac{\sin\theta}{r}\right) \\
=& \frac{1}{2}\left(\frac{\partial(-u_1\sin\theta + u_2\cos\theta)}{\partial r} + \frac{1}{r}\frac{\partial(u_1\cos\theta + u_2\sin\theta)}{\partial\theta} - \frac{1}{r}(-u_1\sin\theta + u_2\cos\theta)\right) \\
=& \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta}{r}\right) .
\end{aligned}$$

In sum, the strain components in polar coordinates can be written as follows:

$$\begin{aligned}
\epsilon_{rr} &= \tfrac{\partial u_r}{\partial r} \\
\epsilon_{\theta\theta} &= \tfrac{1}{r}\tfrac{\partial u_\theta}{\partial \theta} + \tfrac{u_r}{r} \\
\epsilon_{r\theta} = \epsilon_{\theta r} &= \tfrac{1}{2}(\tfrac{\partial u_\theta}{\partial r} + \tfrac{1}{r}\tfrac{\partial u_r}{\partial \theta} - \tfrac{u_\theta}{r})
\end{aligned}$$

The isotropic elastic constitutive relation between strain and stress remains the same in the polar coordinates. In constitutive relation, the only difference is the change of index from $\{i, j\}$ to $\{r, \theta\}$.

The stress tensor $(\sigma_{rr}, \sigma_{r\theta}; \sigma_{\theta r}, \sigma_{\theta\theta})$ can be obtained through the rotation of the Cartesian coordinates by θ, i.e.

$$\begin{aligned}
\sigma_{rr} &= \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + \sigma_{12}\sin 2\theta \\
\sigma_{\theta\theta} &= \sigma_{11}\sin^2\theta + \sigma_{22}\cos^2\theta - \sigma_{12}\sin 2\theta \\
\sigma_{r\theta} &= -\tfrac{\sigma_{11}-\sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta .
\end{aligned}$$

The stress tensor can also be expressed as derivatives of the Airy stress function. See Unit 4 for more details. The equilibrium equations can be also derived through the transformation of coordinates and chain rule as we have done for the strain components. Here, we adopt an alternative tensorial approach and use the results from previous section. In the polar coordinates, $\hat{\mathbf{e}}_r = \cos\theta\hat{\mathbf{e}}_1 + \sin\theta\hat{\mathbf{e}}_2$, and $\hat{\mathbf{e}}_\theta = -\sin\theta\hat{\mathbf{e}}_1 + \cos\theta\hat{\mathbf{e}}_2$, where $\hat{\mathbf{e}}_i$, $i = 1, 2$ are the basis vectors in the Cartesian system. It is straightforward to see that $\frac{\partial\hat{\mathbf{e}}_r}{\partial r} = \frac{\partial\hat{\mathbf{e}}_\theta}{\partial r} = 0$, $\frac{\partial\hat{\mathbf{e}}_r}{\partial\theta} = \hat{\mathbf{e}}_\theta$ and $\frac{\partial\hat{\mathbf{e}}_\theta}{\partial\theta} = -\hat{\mathbf{e}}_r$.

The equilibrium equation in tensorial form reads

$$\begin{aligned}
0 =& \nabla\cdot\boldsymbol{\sigma} \\
=& (\hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \frac{\hat{\mathbf{e}}_\theta}{r}\frac{\partial}{\partial\theta})\cdot(\sigma_{rr}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_r + \sigma_{r\theta}\hat{\mathbf{e}}_r\hat{\mathbf{e}}_\theta + \sigma_{\theta r}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_r + \sigma_{\theta\theta}\hat{\mathbf{e}}_\theta\hat{\mathbf{e}}_\theta) \\
=& \frac{\partial\sigma_{rr}}{\partial r}\hat{\mathbf{e}}_r + \frac{\partial\sigma_{r\theta}}{\partial r}\hat{\mathbf{e}}_\theta + \frac{\sigma_{rr}}{r}\hat{\mathbf{e}}_r + \frac{\sigma_{r\theta}}{r}\hat{\mathbf{e}}_\theta + \frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta}\hat{\mathbf{e}}_r + \frac{\sigma_{\theta r}}{r}\hat{\mathbf{e}}_\theta + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta}\hat{\mathbf{e}}_\theta + \frac{\sigma_{\theta\theta}}{r}(-\hat{\mathbf{e}}_r) \\
=& \left(\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta r}}{\partial\theta} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}\right)\hat{\mathbf{e}}_r + \left(\frac{\partial\sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\sigma_{r\theta}+\sigma_{\theta r}}{r}\right)\hat{\mathbf{e}}_\theta\ .
\end{aligned}$$

Considering the symmetry of the stress tensor, the equilibrium equations in polar coordinates can be written as follows

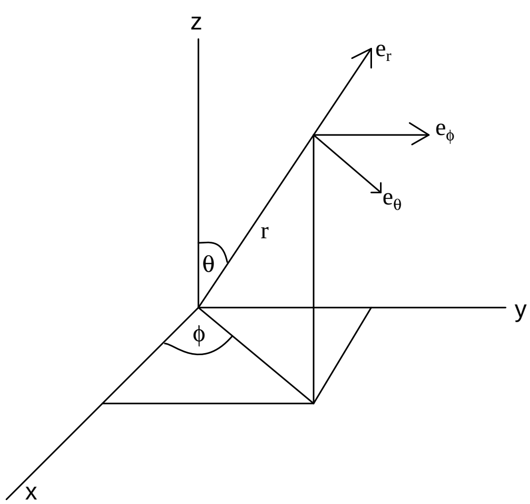
$$\begin{aligned}
\frac{\partial\sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\sigma_{rr}-\sigma_{\theta\theta}}{r} &= 0\ , \\
\frac{1}{r}\frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \frac{\partial\sigma_{r\theta}}{\partial r} + \frac{2\sigma_{r\theta}}{r} &= 0\ .
\end{aligned}$$

B.3.3 Spherical system

In a spherical coordinate system (r, θ, ϕ), the orthonormal basis vectors associated with the coordinates are defined by

$$\begin{aligned}
\hat{\mathbf{e}}_r &= \sin(\theta)\cos(\phi)\hat{\mathbf{e}}_x + \sin(\theta)\sin(\phi)\hat{\mathbf{e}}_y + \cos(\theta)\hat{\mathbf{e}}_z \\
\hat{\mathbf{e}}_\theta &= \cos(\theta)\cos(\phi)\hat{\mathbf{e}}_x + \cos(\theta)\sin(\phi)\hat{\mathbf{e}}_y - \sin(\theta)\hat{\mathbf{e}}_z \\
\hat{\mathbf{e}}_\phi &= -\sin(\phi)\hat{\mathbf{e}}_x + \cos(\phi)\hat{\mathbf{e}}_y
\end{aligned}$$

where $\{\hat{\mathbf{e}}_x, \hat{\mathbf{e}}_y, \hat{\mathbf{e}}_z\}$ are the three basis vectors in the Cartesian coordinate system (x, y, z).



The nabla operator is written as:

$$\nabla = \hat{\mathbf{e}}_r\frac{\partial}{\partial r} + \frac{1}{r}\hat{\mathbf{e}}_\theta\frac{\partial}{\partial\theta} + \frac{1}{r\sin(\theta)}\hat{\mathbf{e}}_\phi\frac{\partial}{\partial\phi}\ .$$

The derivation follows the same way as in the cylindrical coordinates. Main tools are the chain rule of derivative and the fact that (r, θ, ϕ) coordinates can be written as functions of (x, y, z):

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\
\phi &= \arctan \frac{y}{x} \; .
\end{aligned}
$$

The Laplacian operator on a scalar field ψ can be written as

$$
\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 \psi}{\partial \phi^2} \; .
$$