Module 7

General Beam Theory

Learning Objectives

- Review simple beam theory
- Generalize simple beam theory to three dimensions and general cross sections
- Consider combined effects of bending, shear and torsion
- Study the case of shell beams

7.1 Review of simple beam theory

Readings: BC 5 Intro, 5.1

A beam is a structure which has one of its dimensions much larger than the other two. The importance of beam theory in structural mechanics stems from its widespread success in practical applications.

7.1.1 Kinematic assumptions

Readings: BC 5.2

Beam theory is founded on the following two key assumptions known as the Euler-Bernoulli assumptions:

- Cross sections of the beam do not deform in a significant manner under the application of transverse or axial loads and can be assumed as rigid

---

Concept Question 7.1.1. With reference to Figure 7.1,

1. what is the main implication of this assumption on the kinematic description (overall displacement field) of the cross section? ■ **Solution:** The cross section can only undergo a rigid-body motion in its plane, i.e. two rigid body translations and one rotation. ■

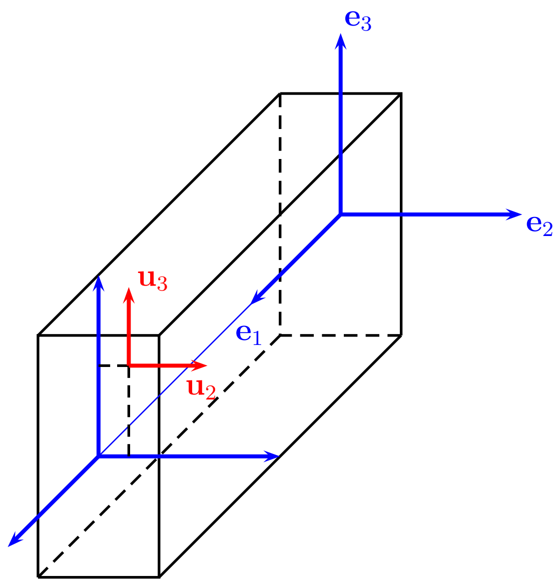

Figure 7.1: First kinematic assumption in Euler-Bernoulli beam theory: rigid in-plane deformation of cross sections.

2. To simplify further the discussion, assume for now that there is no rotation of the cross section around the $\mathbf{e}_3$ axis. Write the most general form of the cross-section in-plane displacement components: ■ **Solution:** The cross section can only translate rigidly in the $\mathbf{e}_2$ and $\mathbf{e}_3$ directions, i.e. the displacement components in the plane cannot depend on the position in the plane x_2, x_3 and:

$$u_2(x_1, x_2, x_3) = \bar{u}_2(x_1),\ u_3(x_1, x_2, x_3) = \bar{u}_3(x_1) \tag{7.1}$$

■

- During deformation, the cross section of the beam is assumed to remain planar and normal to the deformed axis of the beam.

Concept Question 7.1.2. With reference to Figure 7.3,

1. what is the main implication of this assumption on the kinematic description (overall displacement field) of the cross section? ■ **Solution:** The cross section can only translate rigidly in the axial direction, or rotate with respect to the $\mathbf{e}_2$ and $\mathbf{e}_3$ axes by angles θ_2 and θ_3, respectively (using the angles shown in the figure), both angles are of course allowed to be a function of x_1: $\theta_2 = \theta_2(x_1), \theta_3 = \theta_3(x_1)$. ■
2. Based on these kinematic assumptions, write the most general form of the cross-section out-of-plane displacement component: ■ **Solution:** The rigid out-of-plane translation allows for a uniform displacement of the form: $u_1(x_1, x_2, x_3) = \bar{u}_1(x_1)$ as it would be the case for a truss element which sustains

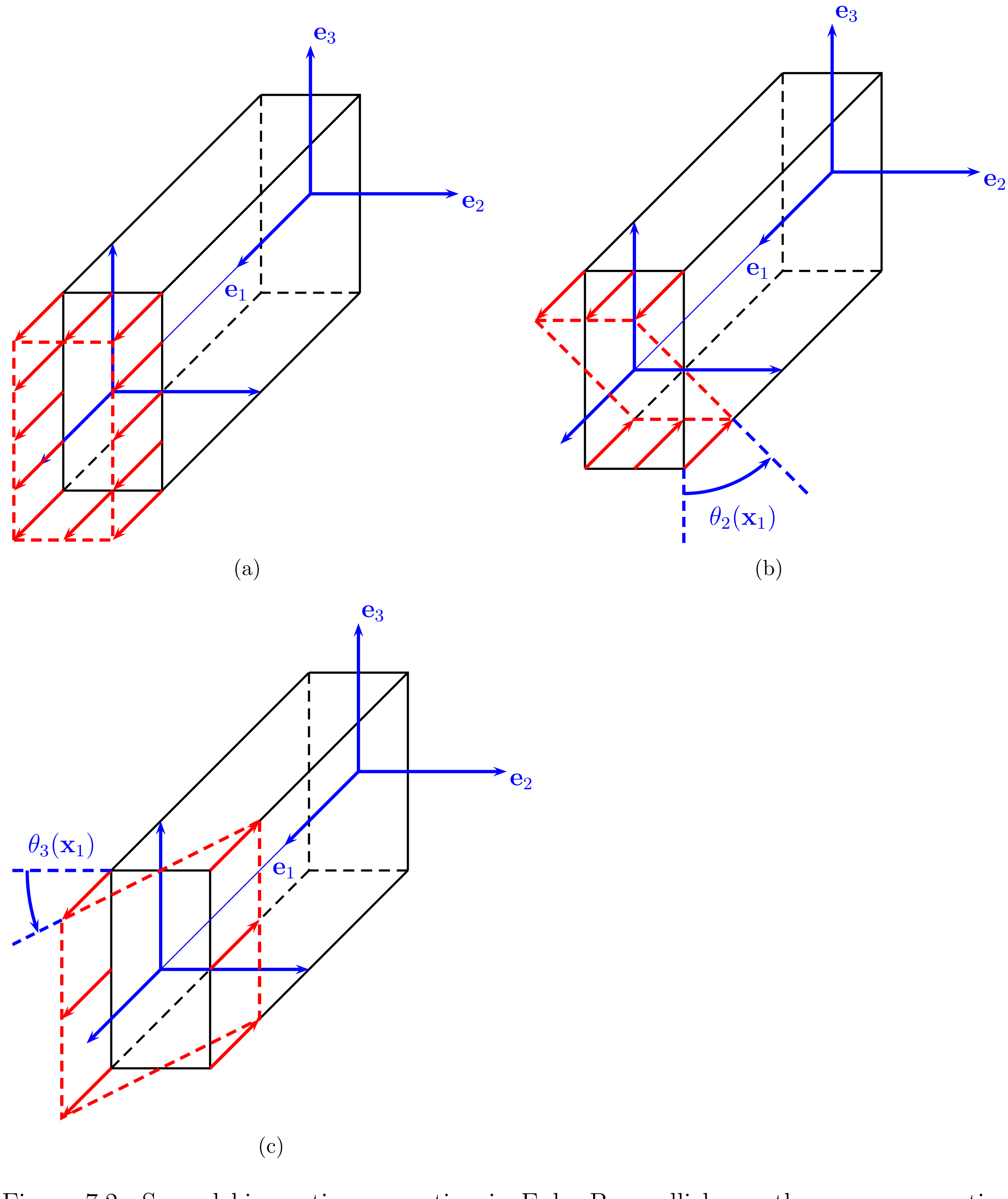

Figure 7.2: Second kinematic assumption in Euler-Bernoulli beam theory: cross sections remain planar after deformation.

a uniform uni-axial strain. The rotations θ_2 and θ_3 will give contributions to the total axial displacement which are linear in the in-plane coordinates x_3 and x_2 respectively, see Figure 7.3. It can be easily inferred from the figure that these contributions adopt the form: $x_3\theta_2(x_1)$ and $-x_2\theta_3(x_1)$, respectively. It should be carefully noted that assuming both positive angles, this contribution indeed has a negative sign.

In summary, the out-of-plane kinematic restrictions imposed by the second Euler-Bernoulli assumption results in the following form of the u_1 displacement component:

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1) + x_3\theta_2(x_1) - x_2\theta_3(x_1) \tag{7.2}$$

It should be noted that we have not defined the origin of the coordinate system but we have implicitly assumed that it corresponds to the intersection of the lines which do not exhibit stretch or contraction under out-of-plane rotations of the cross sections. Later we will define these lines as the *neutral axes* for bending. ■

3. If you noticed, we have not applied the constraint that the sections must remain normal to the deformed axis of the beam. Use this part of the assumption to establish a relation between the displacements fields $\bar{u}_2(x_1), \bar{u}_3(x_1)$ and the angle fields $\theta_2(x_1), \theta_3(x_1)$, respectively, see Figure **??**.

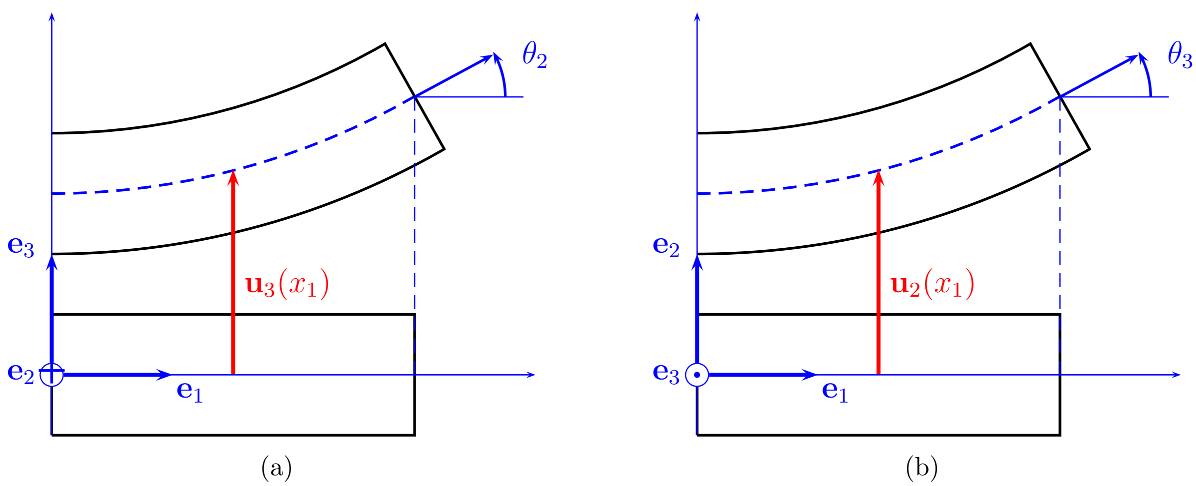

Figure 7.3: Implications of the assumption that cross sections remain normal to the axis of the beam upon deformation.

■ **Solution:** from the figure one can see clearly that:

$$\theta_2 = -\frac{d\bar{u}_3}{dx_1}, \; \theta_3 = \frac{d\bar{u}_2}{dx_1} \tag{7.3}$$

■

Concept Question 7.1.3. Combine the results from all the kinematic assumptions to obtain a final assumed form of the general 3D displacement field in terms of the three unknowns $\bar{u}_1(x_1), \bar{u}_2(x_1), \bar{u}_3(x_1)$: ■ **Solution:**

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1) - x_3\bar{u}_3'(x_1) - x_2\bar{u}_2'(x_1) \tag{7.4}$$

$$u_2(x_1, x_2, x_3) = \bar{u}_2(x_1) \tag{7.5}$$

$$u_3(x_1, x_2, x_3) = \bar{u}_3(x_1) \tag{7.6}$$

■

These assumptions have been extensively confirmed for slender beams made of isotropic materials with solid cross-sections.

Concept Question 7.1.4. Use the kinematic assumptions of Euler-Bernoulli beam theory to derive the general form of the strain field: ■ **Solution:** It follows directly from (7.4) that:

$$\epsilon_{11} = u_{1,1} = \bar{u}_1'(x_1) - x_3\bar{u}_3''(x_1) - x_2\bar{u}_2''(x_1) \tag{7.7}$$

$$\epsilon_{22} = u_{2,2} = \frac{\partial \bar{u}_2(x_1)}{\partial x_2} = 0 \tag{7.8}$$

$$\epsilon_{33} = u_{3,3} = \frac{\partial \bar{u}_3(x_1)}{\partial x_3} = 0 \tag{7.9}$$

$$2\epsilon_{23} = u_{2,3} + u_{3,2} = \frac{\partial \bar{u}_2(x_1)}{\partial x_3} + \frac{\partial \bar{u}_3(x_1)}{\partial x_2} = 0 \tag{7.10}$$

$$2\epsilon_{31} = u_{3,1} + u_{1,3} = \bar{u}_3'(x_1) + \frac{\partial}{\partial x_3}\big(\bar{u}_1(x_1) - x_3\bar{u}_3'(x_1) - x_2\bar{u}_2'(x_2)\big) = \bar{u}_3'(x_1) - \underbrace{\frac{\partial x_3}{\partial x_3}}_{1}\bar{u}_3'(x_1) = 0 \tag{7.11}$$

$$2\epsilon_{12} = u_{1,2} + u_{2,1} = \frac{\partial}{\partial x_2}\big(\bar{u}_1(x_1) - x_3\bar{u}_3'(x_1) - x_2\bar{u}_2'(x_2)\big) + \bar{u}_2'(x_1) = -\underbrace{\frac{\partial x_2}{\partial x_2}}_{1}\bar{u}_2'(x_1) + \bar{u}_2'(x_1) = 0, \tag{7.12}$$

■

Concept Question 7.1.5. It is important to reflect on the nature of the strains due to bending. Interpret the components of the axial strain ϵ_{11} in Euler-Bernoulli beam theory ■ **Solution:**

- The first term represents a uniform strain in the cross section just as those arising in bars subject to uni-axial stress
- The second and third terms tell us that the axial fibers of the beam stretch and contract proportionally to the distance to the neutral axis. The constant of proportionality is the second derivative of the function describing the deflections of the axis of the beam. This can be seen as a linearized version of the local value of the curvature.

- There are no shear strains!!!! This is a direct consequence of assuming that the cross-section remains normal to the deformed axis of the beam.
- There are no strains in the plane. This is a direct consequence of assuming that the cross section is rigid.

■

One of the main conclusions of the Euler-Bernoulli assumptions is that in this particular beam theory the primary unknown variables are the three displacement functions $\bar{u}_1(x_1), \bar{u}_2(x_1), \bar{u}_3(x_1)$ which are only functions of x_1. The full displacement, strain and therefore stress fields do depend on the other independent variables but in a prescribed way that follows directly from the kinematic assumptions and from the equations of elasticity. The purpose of formulating a beam theory is to obtain a description of the problem expressed entirely on variables that depend on a single independent spatial variable x_1 which is the coordinate along the axis of the beam.

7.1.2 Definition of stress resultants

Readings: BC 5.3

Stress resultants are equivalent force systems that represent the integral effect of the internal stresses acting on the cross section. Thus, they eliminate the need to carry over the dependency of the stresses on the spatial coordinates of the cross section x_2, x_3.

The *axial or normal force* is defined by the expression:

$$\boxed{N_1(x_1) = \int_A \sigma_{11}(x_1, x_2, x_3) dA} \tag{7.13}$$

The *transverse shearing forces* are defined by the expressions:

$$\boxed{V_2(x_1) = \int_A \sigma_{12}(x_1, x_2, x_3) dA} \tag{7.14}$$

$$\boxed{V_3(x_1) = \int_A \sigma_{13}(x_1, x_2, x_3) dA} \tag{7.15}$$

$$\tag{7.16}$$

The *bending moments* are defined by the expressions:

$$\boxed{M_2(x_1) = \int_A x_3 \sigma_{11}(x_1, x_2, x_3) dA} \tag{7.17}$$

$$\boxed{M_3(x_1) = -\int_A x_2 \sigma_{11}(x_1, x_2, x_3) dA} \tag{7.18}$$

$$\tag{7.19}$$

The negative sign is needed to generate a positive bending moment with respect to axis $\mathbf{e}_3$, see Figure 7.4

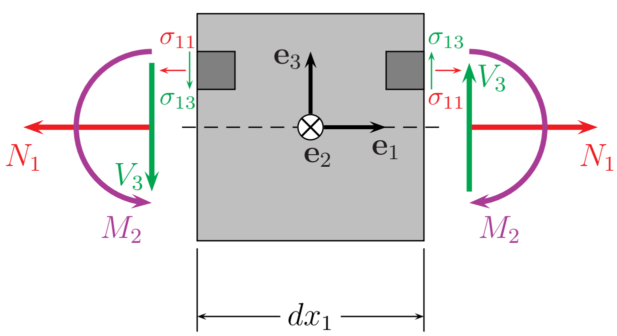

Figure 7.4: Sign conventions for the sectional stress resultants

7.2 Axial loading of beams

Readings: BC 5.4

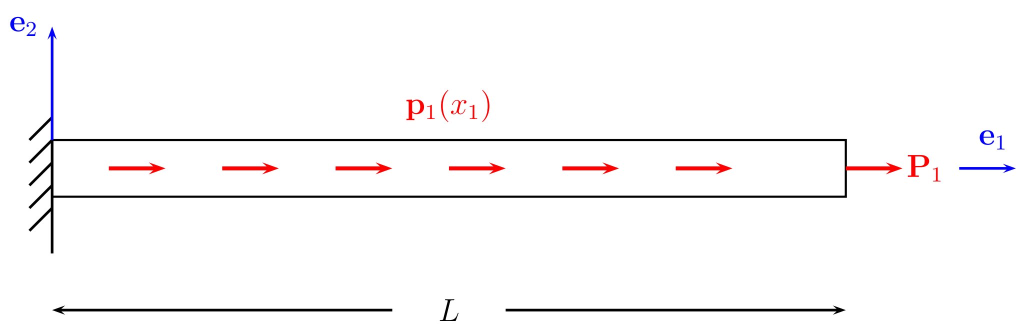

Figure 7.5: Beams subjected to axial loads.

Consider the case where there are no transverse loading on the beam, Figure 7.5. The only external loads possible in this case are either concentrated forces such as the load P_1, or distributed forces per unit length $p_1(x_1)$.

7.2.1 Kinematics

In this case, we will assume that the cross sections will not rotate upon deformation.

Concept Question 7.2.1. Based on this assumption, specialize the general displacement (7.4) and strain field (7.7) to this class of problems and comment on the implications of the possible deformations obtained in this theory ■ **Solution:** Since there are no rotations, $\theta_2 = -\bar{u}_3'(x_1) = 0, \theta_3 = \bar{u}_2'(x_1) = 0$ and we obtain for $u_1(x_1) = \bar{u}_1(x_1)$.

The boundary condition at $x_1 = 0$ dictates $\bar{u}_2(0) = \bar{u}_3(0) = 0$, which combined with the previous assumption of no rotation implies that the transverse deflections are zero everywhere, $\bar{u}_2(x_1) = \bar{u}_3(x_1) = 0$. The displacement field is then:

$$u_1(x_1, x_2, x_3) = \bar{u}_1(x_1)$$
$$u_2(x_1, x_2, x_3) = 0$$
$$u_3(x_1, x_2, x_3) = 0$$

The strain field follows directly from this as:

$$\epsilon_{11}(x_1, x_2, x_3) = \bar{u}_1'(x_1)$$

and all the other strain components are zero. The assumption of allowing only rigid motions of the cross section implies that there cannot be any in-plane strains. This creates a state of uni-axial strain. ■

7.2.2 Constitutive law for the cross section

We will assume a linear-elastic isotropic material and that the transverse stresses $\sigma_{22}, \sigma_{33} \sim 0$. By Hooke's law, the axial stress σ_{11} is given by:

$$\sigma_{11}(x_1, x_2, x_3) = E\epsilon_{11}(x_1, x_2, x_3)$$

Replacing the strain field for this case:

$$\sigma_{11}(x_1, x_2, x_3) = E\bar{u}_1'(x_1) \tag{7.20}$$

In other words, we are assuming a state of uni-axial stress.

Concept Question 7.2.2. This exposes an inconsistency in Euler-Bernoulli beam theory. What is it and how can we justify it? ■ **Solution:** The inconsistency is that we are assuming the kinematics to be uni-axial strain, and the kinetics to be uni-axial stress. In other words one can either have:

$$\epsilon_{22} = \epsilon_{33} = 0$$

(Euler-Bernoulli hypothesis) or

$$\sigma_{22} = \sigma_{33} = 0$$

These two cannot co-exist except when the Poisson ratio is zero. However, this in general has a small effect in most problems of practical interest. The theory is developed assuming that we can ignore both these strains and stresses. ■

The axial force can be obtained by replacing (7.20) in (7.13):

$$N_1(x_1) = \int_{A(x_1)} E\bar{u}_1'(x_1)dA = \underbrace{\int_{A(x_1)} EdA}_{S(x_1)}\,\bar{u}_1'(x_1)$$

We will define:

$$S(x_1) = \int_{A(x_1)} E(x_1, x_2, x_3) dA \tag{7.21}$$

as the *axial stiffness* of the beam, where we allow the Young's modulus to vary freely both in the cross section and along the axis of the beam, and we allow for non-uniform cross section geometries. In the case that the section is homogeneous in the cross section ($E = E(x_1, \cancel{x_2}, \cancel{x_3})$), we obtain: $S(x_1) = E(x_1)A(x_1)$ (This may happen for example in functionally-graded materials). Further, if the section is uniform along x_1 and the material is homogeneous ($E = const$), we obtain: $S = EA$.

We can then write a constitutive relation between the axial force and the appropriate measure of strain for the beam:

$$N_1(x_1) = S(x_1)\bar{u}_1'(x_1) \tag{7.22}$$

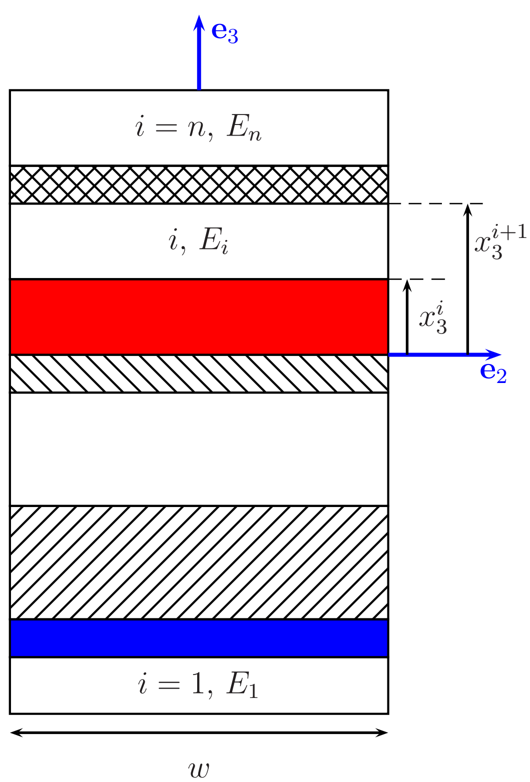

Figure 7.6: Cross section of a composite layered beam.

Concept Question 7.2.3. Axial loading of a composite beam

1. Compute the axial stiffness of a composite beam of width w, which has a uniform cross section with n different layers in direction $\mathbf{e}_3$, where the elastic modulus of layer i is

E^i and its thickness is $t^i = x_3^{i+1} - x_3^i$, as shown in the figure. ■ **Solution:** From equation (7.21),

$$S = \int_{A(x_1)} E(x_1, x_2, x_3) dA = \sum_{i=1}^{n} \int_{A_i} E^i dA^i = \sum_{i=1}^{n} E^i \int_{A_i} dA^i$$

$$= \sum_{i=1}^{n} E^i \underbrace{b\,\overbrace{(x_3^{i+1} - x_3^i)}^{t^i}}_{A_i}$$

Note that we could define an *effective weighted averaged Young's modulus* E^{eff} such that:

$$S = E^{eff} A = \sum_{i=1}^{n} E^i A_i, \Rightarrow E^{eff} = \sum_{i=1}^{n} \frac{A_i}{A} E^i$$

■

2. Compute the stress distribution in the cross section assuming the axial force distribution $N_1(x_1)$ is known: ■ **Solution:** Recalling that the axial strain at x_1 is uniform $\epsilon_{11} = \bar{u}'(x_1)$, and that the stress is given by (7.20), we obtain:

$$\sigma_{11}(x_1, x_2, x_3^i < x_3 < x_3^{i+1}) = E^i \bar{u}'(x_1) = \frac{E^i}{S} N_1(x_1) \tag{7.23}$$

■

3. Interpret the stress distribution obtained. ■ **Solution:** The stress is discontinuous in the cross section. As the section is forced to deform uniformly in the axial direction, layers that are stiffer develop higher stresses ■

Having completed a kinematic and constitutive description, it remains to formulate an appropriate way to enforce equilibrium of beams loaded axially.

7.2.3 Equilibrium equations

For structural elements, we seek to impose equilibrium in terms of resultant forces (rather than at the material point as we did when we derived the equations of stress equilibrium). To this end, we consider the free body diagram of a slice of the beam as shown in Figure 7.7. At x_1 the axial force is $N_1(x_1)$, at $x_1 + dx_1$, $N_1(x_1 + dx_1) = N_1(x_1) + N'(x_1)dx_1$. The distributed force per unit length $p_1(x_1)$ produces a force in the positive x_1 direction equal to $p_1(x_1)dx_1$. Equilibrium of forces in the $\mathbf{e}_1$ direction requires:

$$-N_1(x_1) + p_1(x_1)dx_1 + N_1(x_1) + N'(x_1)dx_1 = 0$$

which implies:

$$\boxed{\frac{dN_1}{dx_1} + p_1 = 0} \tag{7.24}$$

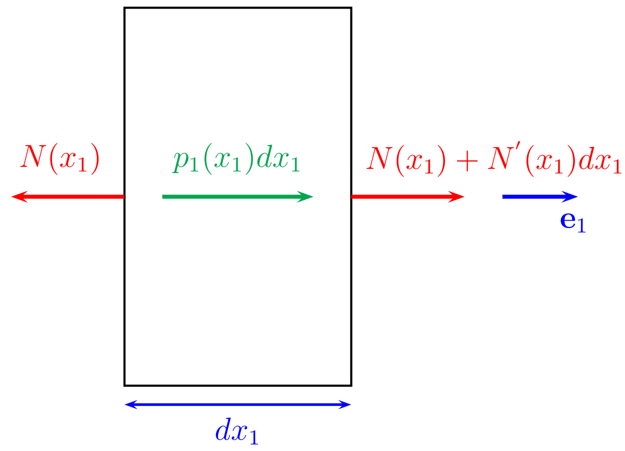

Figure 7.7: Axial forces acting on an infinitesimal beam slice.

7.2.4 Governing equations

Concept Question 7.2.4. 1. Derive a governing differential equation for the axially-loaded beam problem by combining Equations (7.22) and (7.24). ■ **Solution:**

$$\frac{dS(x_1)\bar{u}_1'(x_1)}{dx_1} + p_1 = 0 \tag{7.25}$$

■

2. What type of elasticity formulation does this equation correspond to? ■ **Solution:** It corresponds to a displacement formulation and the equation obtained is a Navier equation. ■

3. What principles does it enforce? ■ **Solution:** It enforces compatibility, the constitutive law and equilibrium. ■

Concept Question 7.2.5. The derived equation requires boundary conditions.

1. How many boundary conditions are required? ■ **Solution:** It's a second order differential equation, thus it requires two boundary conditions ■

2. What type of physical boundary conditions make sense for this problem and how are they expressed mathematically? ■ **Solution:** The bar can be

   - fixed, this implies that the displacement is specified to be zero

   $$\bar{u}_1 = 0$$

   - free (unloaded), which implies that:

   $$N_1 = S\bar{u}_1' = 0, \Rightarrow \bar{u}_1' = 0$$

- subjected to a concentrated load P_1, which implies that:

$$N_1 = S\bar{u}_1' = P_1$$

■

This completes the formulation for axially-loaded beams.

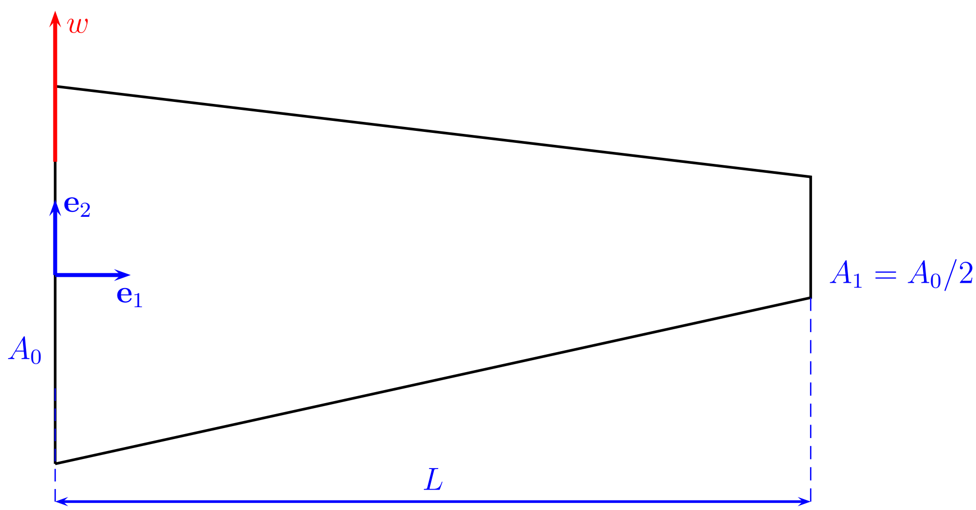

Figure 7.8: Schematic of a helicopter blade rotating at an angular speed ω

Concept Question 7.2.6. *Helicopter blade under centrifugal load* A helicopter blade of length $L = 5m$ is rotating at an angular velocity $\omega = 600$rpm about the $\mathbf{e}_2$ axis. The blade is made of a carbon-fiber reinforced polymer (CFRP) composite with mass density $\rho = 1500\text{kg} \cdot \text{m}^{-3}$, a Young's modulus $E = 80GPa$ and a yield stress $\sigma_y = 50$MPa. The area of the cross-section of the blade decreases linearly from a value $A_0 = 100\text{cm}^2$ at the root to $A_1 = A_0/2 = 50\text{cm}^2$ at the tip.

1. give the expression of the distributed axial load corresponding to the centrifugal force ■ **Solution:** The area can then be written as

$$A(x_1) = A_0 + (A_1 - A_0)\frac{x_1}{L} = A_0\big(1 - \tfrac{x_1}{2L}\big) = 10^{-2}\text{m}^2\big(1 - \frac{x_1}{10\text{m}}\big)$$

$$p_1(x_1) = \rho A(x_1)\omega^2 x_1 = 6000\pi^2\big(1 - \frac{x_1}{10\text{m}}\big)x_1 \cdot \text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-2}$$

■

2. Integrate the equilibrium equation (7.24) and apply appropriate boundary conditions to obtain the axial force distribution $N_1(x_1)$ in the blade. ■ **Solution:**

$$N_1'(x_1) + \rho\Big[A_0\big(1 - \tfrac{x_1}{2L}\big)\Big]\omega^2 x_1 = 0$$

$$N_1(x_1) = \rho\omega^2 A_0 x_1^2\Big(\frac{x_1}{6L} - \frac{1}{2}\Big) + C$$

The boundary condition is: $N_1(L) = 0$, let's go ahead and apply it:

$$0 = \rho\omega^2 A_0 L^2 \underbrace{\Big(\frac{\cancel{L}}{6\cancel{L}} - \frac{1}{2}\Big)}_{-1/3} + C \ \Rightarrow C = \frac{1}{3}\rho\omega^2 A_0 L^2$$

Replacing in the previous expression:

$$N_1(x_1) = \frac{1}{6}\rho\omega^2 L^2 A_0(\eta^3 - 3\eta^2 + 2\Big)$$

where we defined $\eta = x_1/L$. ■

3. What is the maximum axial force and where does it happen? ■ **Solution:** The maximum in the range $0 \leq \eta \leq 1$ is at $\eta = 0$. The maximum happens at $x_1 = 0$ and the value is: $N_1^{max} = N_1(0) = \frac{1}{3}\rho\omega^2 L^2 A_0 = 685389$N ■

4. Provide an expression for the axial stress distribution $\sigma_{11}(x_1)$ ■ **Solution:**

$$\sigma_{11}(x_1) = \frac{N_1(x_1)}{A(x_1)} = \frac{\frac{1}{6}\rho\omega^2 L^2 \cancel{A_0}(\eta^3 - 3\eta^2 + 2\Big)}{\cancel{A_0}\big(1 - \frac{\eta}{2}\big)}$$

■

5. What is the maximum stress, where does it happen, does the material yield? ■ **Solution:** For the values given, one can find the maximum to be $\sigma_{11}^{max} = 36MPa$ and it happens at $x_1 = 0.194L$. There is no yielding as $\sigma_{11}^{max} < \sigma_y$ and we are considering uni-axial stress. ■

6. The displacement can be obtained by integrating the strain:

$$\epsilon_{11} = \bar{u}_1'(x_1) = \frac{\sigma_{11}(x_1)}{E}$$

and applying the boundary condition $\bar{u}_1(0) = 0$. The solution can be readily found to be:

$$\bar{u}_1(x_1) = \frac{\rho\omega^2 L^3}{3E}\Big[2\eta + \frac{\eta^2}{2} - \frac{\eta^3}{3} + 2\log\big(1 - \frac{\eta}{2}\big)\Big]$$

Verify that the correct strain is obtained and that the boundary condition is satisfied: ■ **Solution:** The boundary condition is readily verified. The strain is:

$$\bar{u}_1'(x_1) = \frac{\rho\omega^2 L^3}{3E}\Big[2 + \eta - \eta^2 - \frac{1}{1 - \frac{\eta}{2}}\Big]$$

which is the same as we obtained above. ■

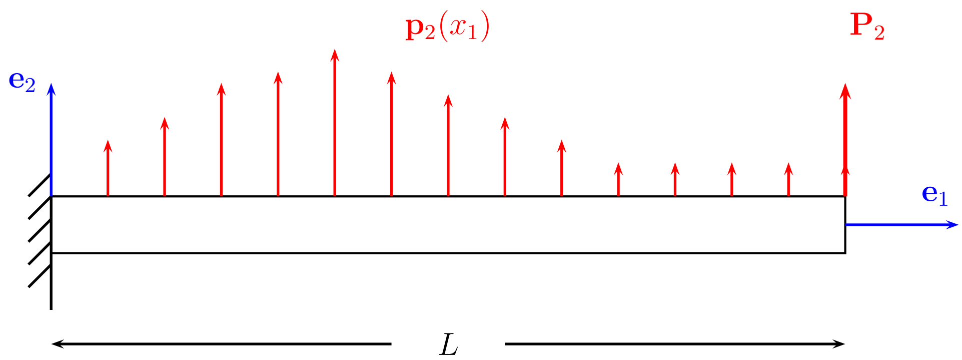

Figure 7.9: Beam subjected to tranverse loads

7.3 Beam bending

Readings: BC 5.5

Beams have the defining characteristic that they can resist loads acting transversely to its axis, Figure 7.9 by bending or deflecting outside of their axis. This bending deformation causes internal axial and shear stresses which can be described by equipolent stress resultant moments and shearing forces.

We will start by analyzing beam bending in the plane $\mathbf{e}_1, \mathbf{e}_2$. Combined bending in different planes can be treated later by using the principle of superposition.

The Euler-Bernoulli kinematic hypothesis (7.4) reduces in this case to

$$u_1(x_1, x_2, x_3) = -x_2\bar{u}_2'(x_1), \; u_2(x_1, x_2, x_3) = \bar{u}_2(x_1), \; u_3(x_1, x_2, x_3) = 0$$

The strains to:

$$\epsilon_{11}(x_1, x_2, x_3) = u_{1,1} = -x_2\bar{u}_2''(x_1)$$

7.3.1 Constitutive law for the cross section

Hooke's law reduces one more time to:

$$\sigma_{11}(x_1, x_2, x_3) = E\epsilon_{11}(x_1, x_2, x_3) = -Ex_2\bar{u}_2''(x_1) \tag{7.26}$$

Concept Question 7.3.1. Assuming E is constant in the cross section, comment on the form of the stress distribution ■ **Solution:** it is readily seen that a linear stress distribution through the thickness is linear in x_2. ■

We now proceed to compute the stress resultants of section 7.1.2.

Concept Question 7.3.2. *Location of the neutral axis:* We will see in this question that the x_2 location of the fibers that do not stretch in the $\mathbf{e}_1$ direction, which is where we are going to place our origin of the x_2 coordinates is determined by the requirement of axial equilibrium of internal stresses.

1. The only applied external forces are in the $\mathbf{e}_2$ direction. Based on this, what can you say about the axial force $N_1(x_1)$?

   ■ **Solution:** From force equilibrium in direction $\mathbf{e}_1$ we conclude that the axial force must vanish at all cross sections x_1, i.e. $N_1(x_1) = 0, \forall x_1$ ■

2. Write the expression for the axial force ■ **Solution:** From equation (7.13)

$$N_1(x_1) = \int_{A(x_1)} \sigma_{11}(x_1, x_2, x_3) dA = \int_{A(x_1)} (-1) E x_2 \bar{u}_2''(x_1) dA = (-1) \left[\int_{A(x_1)} E x_2 dA \right] \bar{u}_2''(x_1)$$

   ■

3. From here, obtain an expression that enforces equilibrium in the $\mathbf{e}_1$ direction. Interpret its meaning ■ **Solution:** Setting the axial force to zero, and noticing that the curvature $\bar{u}_2''(x_1) \neq 0$, we obtain that the square bracket in the previous relation must vanish:

$$\int_{A(x_1)} E x_2 dA = 0$$

   This can be interpreted as a *modulus-weighted first moment of area* of the cross section. For uniform E, we obtain simply: $\int_{A(x_1)} x_2 dA = 0$. ■

4. Define the *modulus-weighted centroid* of the cross section by the condition:

$$S(x_1) x_2^c(x_1) = \int_{A(x_1)} E x_2 dA, \Rightarrow \boxed{x_2^c(x_1) = \frac{1}{S(x_1)} \int_{A(x_1)} E x_2 dA}$$

   and interpret the meaning of using x_2^c as the origin of the coordinate system ■ **Solution:** it can be seen that setting the zero for x_2 at x_2^c, the modulus-weighted first moment of area of the cross section vanishes. ■

5. Compare the location of the modulus-weighted centroid, the center of mass and the center of area for a general cross section and for a homogeneous one ■ **Solution:**

$$x_2^c = \frac{\int_A E x_2 dA}{\int_{A(x_1)} E dA}, \; x_2^m = \frac{\int_A \rho x_2 dA}{\int_a \rho dA}, \; x_2^a = \frac{\int_A x_2 dA}{\int_A dA}$$

   These locations don't match in general. For homogeneous cross sections:

$$x_2^c = \frac{\cancel{E} \int_A x_2 dA}{\cancel{E} \int_A dA} = x_2^m = \frac{\cancel{\rho} \int_A x_2 dA}{\cancel{\rho} \int_A dA} = x_2^a = \frac{\int_A x_2 dA}{\int_A dA}$$

   , they do. ■

We now consider the internal bending moment.

Concept Question 7.3.3. Specialize the definition of the M_3 stress resultant (7.17)

$$M_3(x_1) = -\int_{A(x_1)} x_2\sigma_{11}(x_1, x_2, x_3)dA$$

to the case under consideration by using the stress distribution resulting from the Euler-Bernoulli hypothesis, $\sigma_{11}(x_1, x_2, x_3) = -Ex_2\bar{u}_2''(x_1)$ to obtain a relation between the bending moment and the local curvature $\bar{u}_2''(x_1)$.

■ **Solution:** By direct substitution and some algebraic manipulation we obtain:

$$M_3(x_1) = \cancel{-}\int_{A(x_1)} x_2(\cancel{-1})Ex_2\bar{u}_2''(x_1)dA = \underbrace{\left[\int_{A(x_1)} Ex_2^2 dA\right]}_{H_{33}^c(x_1)} \bar{u}_2''(x_1)$$

■

We can see that we obtain a linear relation between the bending moment and the local curvature (*moment-curvature relationship*):

$$\boxed{M_3(x_1) = H_{33}^c(x_1)\bar{u}_2''(x_1)} \tag{7.27}$$

The constant of proportionality will be referred to as the *centroidal bending stiffness* (also sometimes known as the *flexural rigidity*):

$$\boxed{H_{33}^c(x_1) = \int_{A(x_1)} Ex_2^2 dA} \tag{7.28}$$

In the case of a homogeneous cross section of Young's modulus $E(x_1)$:

$$H_{33}^c(x_1) = E(x_1)\underbrace{\int_{A(x_1)} x_2^2 dA}_{I_{33}}$$

we obtain the familiar:

$$M = EI\bar{u}_2''(x_1)$$

Concept Question 7.3.4. *Modulus-weighted centroid, Bending stiffness and bending stress distribution in a layered composite beam* A composite beam of width b, which has a uniform cross section with n different layers in direction $\mathbf{e}_2$, where the elastic modulus of layer i is E^i and its thickness is $t^i = x_2^{i+1} - x_2^i$, as shown in Figure 7.10.

1. Compute the position of the modulus-weighted centroid ■ **Solution:** The modulus weighted centroid is obtained by computing the modulus weighted first moment of area:

$$\int_A Ex_2 dA = \sum_{i=1}^n E^i \int_{A^i} x_2 dA = \sum_{i=1}^n E^i b \int_{x_2^i}^{x_2^i+t^i} x_2 dA$$

$$= \sum_{i=1}^n E^i b\frac{1}{2}(\cancel{(x_2^i)^2} + 2x_2^i t^i + (t^i)^2 - \cancel{(x_2^i)^2}) = \sum_{i=1}^n E^i A^i (x_2^i + \frac{t^i}{2})$$

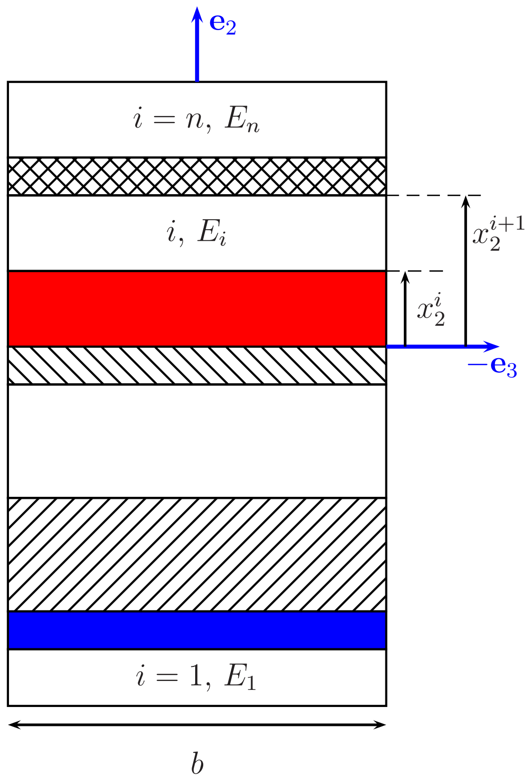

Figure 7.10: Cross section of a composite layered beam.

and dividing by the modulus weighted area (or axial stiffness):

$$S = \int_A E dA = \sum_{i=1}^{n} E^i A^i$$

$$\Rightarrow \boxed{x_2^c = \frac{\sum_{i=1}^{n} E^i A^i (x_2^i + \frac{t^i}{2})}{\sum_{i=1}^{n} E^i A^i}}$$

■

2. Compute the bending stiffness ■ **Solution:** From equation (7.28),

$$H_{33}^c = \int_{A(x_1)} E(x_1, x_2, x_3) x_2^2 dA = \sum_{i=1}^{n} \int_{A_i} E^i x_2^2 dA^i = \sum_{i=1}^{n} E^i \int_{A_i} x_2^2 dA^i$$
$$= \sum_{i=1}^{n} E^i b \frac{1}{3} \left[\left(x_2^{i+1}\right)^3 - \left(x_2^i\right)^3 \right]$$

where care should be exercised to measure the distance x_2 from the location of the modulus weighted centroid x_2^c. ■

3. Compute the σ_{11} stress distribution in the cross section assuming the bending moment

M_3 is known: ■ **Solution:** Recalling (7.34), we obtain:

$$\sigma_{11}(x_1, x_2^i < x_2 < x_2^{i+1}, x_3) = E^i \epsilon_{11}(x_1, x_2, x_3) = -E^i x_2 \bar{u}_2''(x_1) = -E^i x_2 \frac{M_3}{H_{33}^c}$$

■

4. Interpret the stress distribution obtained. ■ **Solution:** The axial strain distribution is linear over the cross section, this forces the stress distribution to be piecewise linear within each layer of different elastic modulus, but discontinuous at interlayer boundaries. ■

5. Specialize to the case that the section is homogeneous with Young's modulus E: ■ **Solution:** In this case, $H_{33}^c = EI_{33}$, and the previous equation becomes the familiar formula for simple beam theory:

$$\sigma_{11}(x_1, x_2, x_3) = -\cancel{E} x_2 \frac{M_3}{\cancel{E} I_{33}}, \Rightarrow \boxed{\sigma_{11}(x_1, x_2, x_3) = -x_2 \frac{M_3(x_1)}{I_{33}}} \tag{7.29}$$

■

Concept Question 7.3.5. *Bi-material cross section properties*

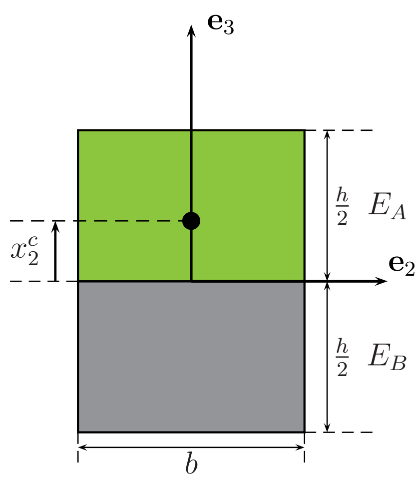

Figure 7.11: Bi-material beam

For the cantilevered beam shown in Figure 7.11,

1. Compute the location of the modulus-weighted centroid: ■ **Solution:**

$$x_2^c = x_2^c = \frac{\sum_{i=1}^n E^i A^i (x_2^i + \frac{t^i}{2})}{\sum_{i=1}^n E^i A^i}$$

■

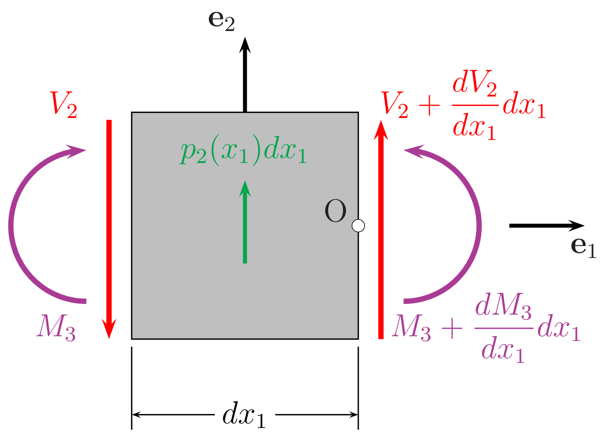

Figure 7.12: Equilibrium of a beam slice subjected to transverse loads

7.3.2 Equilibrium equations

Concept Question 7.3.6. Consider the equilibrium of a slice of a beam subjected to transverse loads. Using the figure,

1. write the equation of equilibrium of forces in the $\mathbf{e}_2$ direction and then derive a differential equation relating the shear force $V_2(x_1)$ and the distributed external force $p_2(x_1)$. ■ **Solution:**

$$\cancel{-V_2(x_1)} + p_2(x_1)dx_1 + \cancel{V_2(x_1)} + V_2'(x_1)dx_1 = 0,$$

$$\Rightarrow \boxed{V_2'(x_1) + p_2(x_1) = 0} \quad (7.30)$$

■

2. do the same for equilibrium of moments in the $\mathbf{e}_3$ axis about point **O**. ■ **Solution:**

$$\cancel{-M_3(x_1)} + V_2(x_1)dx_1 - \cancel{p_2(x_1)dx_1\frac{dx_1}{2}}^{\,h.o.t.} + \cancel{M_3(x_1)} + M_3'(x_1)dx_1 = 0$$

$$\Rightarrow \boxed{M_3'(x_1) + V_2(x_1) = 0} \quad (7.31)$$

■

3. Eliminate the Shear force from the previous two equations to obtain a single equilibrium equation relating the bending moment and the applied distributed load: ■ **Solution:** Differentiate the second $\big(M_3'(x_1) + V_2(x_1)\big)' = M_3''(x_1) + V_2'(x_1) = 0$ and replace in the first to obtain:

$$-M_3''(x_1) + p_2(x_1) = 0$$

$$\Rightarrow \boxed{M_3''(x_1) = p_2(x_1)} \quad (7.32)$$

■

7.3.3 Governing equations

Concept Question 7.3.7. 1. Derive a governing differential equation for the transversely-loaded beam problem by combining Equations (7.27) and (7.32). ■ **Solution:**

$$\boxed{\left(H_{33}(x_1)\bar{u}_2''(x_1)\right)'' = p_2(x_1)} \tag{7.33}$$

■

2. What type of elasticity formulation does this equation correspond to? ■ **Solution:** It corresponds to a displacement formulation and the equation obtained is a Navier equation. ■

3. What principles does it enforce? ■ **Solution:** It enforces compatibility, the constitutive law and equilibrium. ■

Concept Question 7.3.8. The equation requires four boundary conditions since it is a fourth-order differential equation. Express the following typical boundary conditions mathematically

1. clamped at one end ■ **Solution:** this implies that the deflection and the rotation are restricted at that point

$$\bar{u}_2 = 0, \bar{u}_2' = 0$$

■

2. simply supported or pinned ■ **Solution:** this implies that the deflection is restricted but the the slope is arbitrary. The freedom to rotate implies that the support cannot support a bending moment reaction and $M_3 = 0$ at that point

$$\bar{u}_2 = 0, M_3 = H_{33}^c \bar{u}_2'' = 0, \Rightarrow u_2'' = 0$$

■

3. free (unloaded) ■ **Solution:** implies that both the bending moment and the shear force must vanish:

$$u_2'' = 0, V_2 = -M_3' = -(H_{33}^c \bar{u}_2'')' = 0$$

■

4. subjected to a concentrated transverse load P_2 ■ **Solution:** this implies that that the bending moment must vanish but the shear force must equal the applied load

$$u_2'' = 0, V_2 = -M_3' = -(H_{33}^c \bar{u}_2'')' = P_2$$

■

Concept Question 7.3.9. *Cantilevered beam under uniformly distributed transverse load*

A cantilevered beam (clamped at $x_1 = 0$ and free at $x_1 = L$) is subjected to a uniform load per unit length gp_0.

1. Specialize the general beam equation to this problem ■ **Solution:** The governing equation for $\bar{u}_2(x_1)$ reads
$$H_{33}^c \bar{u}_2^{IV} = p_0$$
as the bending stiffness is constant. ■

2. Write down the boundary conditions for this problem:

   ■ **Solution:** At the clamped end $x_1 = 0$, the transverse displacement and rotation of the section (slope of the beam) are both zeros, i.e.
$$\bar{u}_2(0) = \bar{u}_2'(0) = 0.$$
At the free end $x_1 = L$, the bending moment and the shear force are zero, i.e.
$$H_{33}^c \bar{u}_2''(L) = 0, \; -H_{33}^c \bar{u}_2'''(L) = 0$$
■

3. Integrate the governing equation and apply the boundary conditions to obtain the deflection $\bar{u}_2(x_1)$, bending moment $M_3(x_1)$ and shear force $V_2(x_1)$. ■ **Solution:** The governing equation can be integrated directly and the the following general expression for the deflection is obtained: After the first integration we get:
$$H_{33}^c \bar{u}_2''' = p_0 x_1 + c_1 = -V_2(x_1)$$
After the second
$$H_{33}^c \bar{u}_2'' = \frac{p_0}{2} x_1^2 + c_1 x_1 + c_2 = M_3(x_1)$$
After the third
$$H_{33}^c \bar{u}_2' = \frac{p_0}{6} x_1^3 + \frac{c_1}{2} x_1^2 + c_2 x_1 + c_3$$
After the last
$$H_{33}^c \bar{u}_2(x_1) = \frac{1}{24} p_0 x_1^4 + \frac{c_1}{6} x_1^3 + \frac{c_2}{2} x_1^2 + c_3 x_1 + c_4$$
The clamped boundary condition at $x_1 = 0$ implies: $\bar{u}_2(0) = c_4 = 0, \bar{u}_2'(0) = c_3 = 0$. The free boundary condition at $x_1 = L$ implies: $V_2(L) = p_0 L + c_1 = 0, \Rightarrow c_1 = -p_0 L$, and $M_3(L) = \frac{p_0}{2} L^2 + \underbrace{(-p_0 L)}_{c_1} L + c_2 = 0, \Rightarrow c_2 = \frac{p_0}{2} L^2$. Replacing in the expressions above :
$$\boxed{V_2(x_1) = p_0 L \big(1 - \frac{x_1}{L}\big)}$$

$$M_3(x_1) = \frac{p_0}{2}x_1^2 + \underbrace{(-p_0L)}_{c_1}x_1 + \underbrace{\frac{p_0}{2}L^2}_{c_2} = \frac{p_0}{2}L^2\big[1 - 2\frac{x_1}{L} + \big(\frac{x_1}{L}\big)^2\big]$$

$$\boxed{M_3(x_1) = \frac{p_0}{2}L^2\big[1 - \big(\frac{x_1}{L}\big)\big]^2}$$

$$H_{33}^c\bar{u}_2(x_1) = \frac{1}{24}p_0x_1^4 + \frac{\overbrace{(-p_0L)}^{c_1}}{6}x_1^3 + \frac{\overbrace{\frac{p_0}{2}L^2}^{c_2}}{2}x_1^2$$

$$\boxed{\bar{u}_2(x_1) = \frac{p_0L^4}{24H_{33}^c}\big(\frac{x_1}{L}\big)^2\Big[\big(\frac{x_1}{L}\big)^2 - 4\big(\frac{x_1}{L}\big) + 6\Big]}$$

■

4. Compute the maximum deflection, maximum moment and maximum σ_{11} stress (for the case of a solid rectangular wing spar of length $L = 1$m, width $b = 5$mm, height $h = 3$cm, and Young's modulus $E = 100GPa$ when the load is $p_0 = 10N/m$. ■
Solution: In this case, the stiffness $H_{33}^c = EI = \frac{Ebh^3}{12} = 200\text{N}\cdot\text{m}^2$ and their locations: The maximum deflection clearly occurs at the free end and takes the value

$$\bar{u}_2^{max} = \bar{u}_2(L) = \frac{p_0L^4}{8EI} = 3.1\text{cm}.$$

The maximum moment clearly occurs at the root of the cantilever $x_1 = 0$ and takes the value:

$$M_3^{max} = M_3(L) = \frac{p_0}{2}L^2 = 11.25\text{N}\cdot\text{m}.$$

The maximum stress can then be obtained from equation (7.29)

$$\sigma_{11}^{max} = \sigma_{11}(0, \pm\frac{h}{2}, x_3) = \mp\frac{h}{2}\frac{M_3(x_1)}{I_{33}} = \mp 56.25\text{MPa}$$

■

7.4 Beams loaded by transverse loads in general directions

Readings: BC 6

So far we have considered beams of fairly simple cross sections (e.g. having symmetry planes which are orthogonal) and transverse loads acting on the planes of symmetry. Figure 7.13 shows examples of beams loaded on a plane which does not coincide with a plane of symmetry of its cross section.

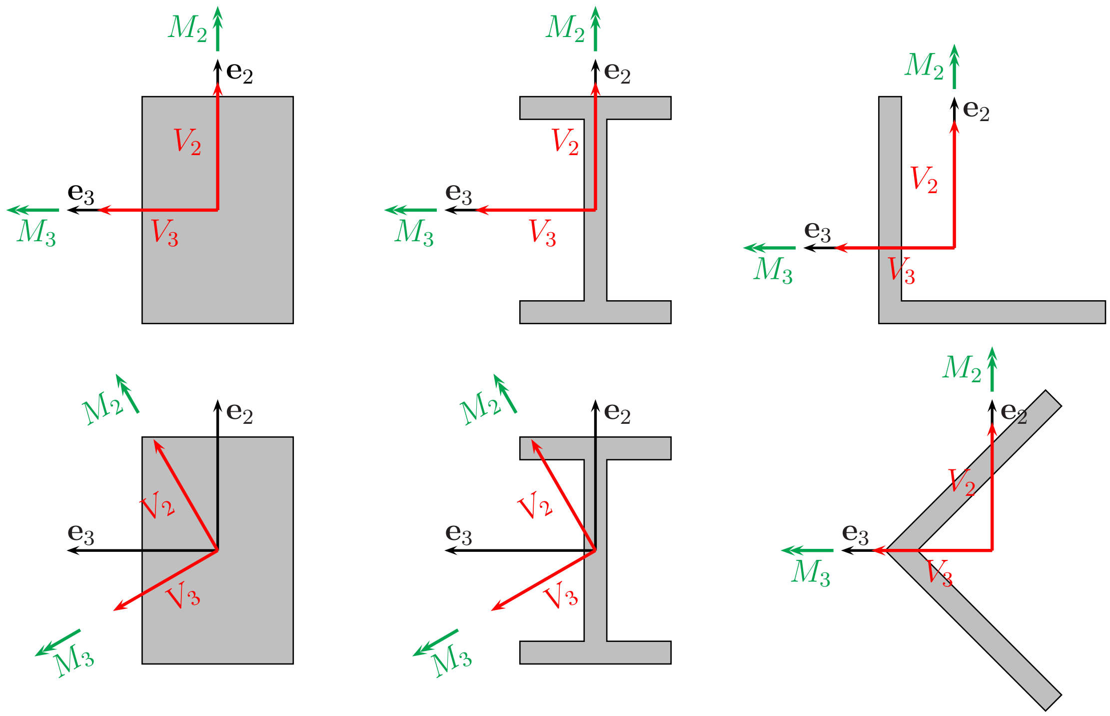

Figure 7.13: Loading of beams in general planes and somewhat general cross sections

In this section, we will consider beams with cross section of arbitrary shape which are loaded on planes that do not in general coincide with symmetry planes (or as we will see later more precisely, with principal directions of inertia of the cross section).

We will still adopt Euler-Bernoulli hypothesis, which implies that the kinematic assumptions about the allowed deformation modes of the beam remain the same, see Section 7.1.1.

The displacement field is still given by equations (7.4), whereas the strain field is given by equations (7.7). It should be noted that the origin of coordinates in the cross section is still unspecified.

7.4.1 Constitutive law for the cross section

We will assume that the beam is made of linear elastic isotropic materials and use Hooke's law. Since the strain distribution is still bound by the sames constraints, the stress distribution will be as before:

$$\sigma_{11}(x_1, x_2, x_3) = E\epsilon_{11}(x_1, x_2, x_3) = E[\bar{u}_1'(x_1) - x_2\bar{u}_2''(x_1) - x_3\bar{u}_3''(x_1)] \tag{7.34}$$

Following with the by now usual plan to build a structural theory, we proceed to compute the resultants:

Axial force N_1

$$
\begin{aligned}
N_1(x_1) &= \int_A \sigma_{11}(x_1, x_2, x_3) dA \\
&= \underbrace{\left[\int_A E dA\right]}_{S} \bar{u}_1'(x_1) - \underbrace{\left[\int_A E x_2 dA\right]}_{S_2} \bar{u}_2''(x_1) - \underbrace{\left[\int_A E x_3 dA\right]}_{S_3} \bar{u}_3''(x_1)
\end{aligned}
$$

$$
\boxed{N_1(x_1) = S\bar{u}_1'(x_1) - S_2\bar{u}_2''(x_1) - S_3\bar{u}_3''(x_1)} \quad (7.35)
$$

where S is the modulus-weighted area or axial stiffness, S_2, S_3 are respectively the modulus-weighted first moments of area of the cross section with respect to the $\mathbf{e}_3$ and $\mathbf{e}_2$ axes.

Bending moments $M_2(x_1), M_3(x_1)$

$$
\begin{aligned}
M_2(x_1) &= \int_A \sigma_{11} x_3 dA = \\
&= \underbrace{\left[\int_A E x_3 dA\right]}_{S_3} \bar{u}_1'(x_1) - \underbrace{\left[\int_A E x_2 x_3 dA\right]}_{H_{23}} \bar{u}_2''(x_1) - \underbrace{\left[\int_A E x_3^2 dA\right]}_{H_{22}} \bar{u}_3''(x_1)
\end{aligned}
$$

$$
\boxed{M_2(x_1) = S_3\bar{u}_1'(x_1) - H_{23}\bar{u}_2''(x_1) - H_{22}\bar{u}_3''(x_1)} \quad (7.36)
$$

$$
\begin{aligned}
M_3(x_1) &= -\int_A \sigma_{11} x_2 dA = \\
&= -\underbrace{\left[\int_A E x_2 dA\right]}_{S_2} \bar{u}_1'(x_1) + \underbrace{\left[\int_A E x_2^2 dA\right]}_{H_{33}} \bar{u}_2''(x_1) + \underbrace{\left[\int_A E x_3 x_2 dA\right]}_{H_{23}} \bar{u}_3''(x_1)
\end{aligned}
$$

$$
\boxed{M_3(x_1) = -S_2\bar{u}_1'(x_1) + H_{33}\bar{u}_2''(x_1) + H_{23}\bar{u}_3''(x_1)} \quad (7.37)
$$

We note that we have used some of the previously defined section stiffness coefficients S, H_{33}, but we have also introduced some new ones. Summarizing all:

Area:	$S = \int_A E dA$
First moment of area wrt $\mathbf{e}_3$	$S_2 = \int_A E x_2 dA$
First moment of area wrt $\mathbf{e}_2$	$S_3 = \int_A E x_3 dA$
Second moment of area wrt $\mathbf{e}_3$	$H_{22} = \int_A E x_3^2 dA$
Second moment of area wrt $\mathbf{e}_2$	$H_{33} = \int_A E x_2^2 dA$
Second cross moment of area wrt $\mathbf{e}_2, \mathbf{e}_3$	$H_{23} = \int_A E x_2 x_3 dA$

Table 7.1: Modulus-weighted cross section stiffness coefficients

---

Concept Question 7.4.1. Give an interpretation to the various cross section stiffness coefficients by observing the "strains" and resultant forces they relate ■ **Solution:**

- S is the direct stiffness for axial deformation, i.e. it determines what axial force is produced per unit axial deformation.

- S_2 is the cross stiffness between curvature in the $\mathbf{e}_3$ direction (12-plane) and the axial force, i.e. it determines what axial force is produced per unit curvature in that plane. Conversely, it is the cross stiffness determining the moment M_3 produced per unit axial strain.

- S_3 similar discussion to S_2

- H_{22}, H_{33} are the direct stiffnesses relating section bending strain measure (curvatures) and corresponding bending moment.

- H_{23} is the cross stiffness relating curvature in one plane with moment in the other.

These conclusions also apply in inverse form, i.e. by inverting these relations we obtain coefficients that determine the "sectional strain measure" produced per unit resultant force, e.g. the curvature in a given plane produced per unit axial force or moments in either plane, etc. ■

The main conclusion from this general beam theory is that there is a coupling among all stress resultants and all "strain measures". Specifically, this means that a curvature in one plane can cause not only a bending moment in the respective plane but also a moment in the plane orthogonal to it as well as an axial force. Also, that the axial strain $\bar{u}_1'$ can cause moments in both orthogonal planes.

A first simplification of these expressions is obtained if we first find the *modulus-weighted centroid* of the cross section x_2^c, x_3^c and then refer all our quantities with respect to that point (i.e. place the origin of our axes from where we measure x_2, x_3 at that point). In that case, as we saw before:

$$x_2^c = \frac{\overbrace{\int_A E x_2 dA}^{S_2}}{\underbrace{\int_A E dA}_{S}} = 0, x_3^c = \frac{\overbrace{\int_A E x_3 dA}^{S_3}}{\underbrace{\int_A E dA}_{S}} = 0, \tag{7.38}$$

and the coupling between axial and flexural quantities disappears, i.e. the sectional constitutive equations become:

$$\boxed{N_1(x_1) = S\bar{u}_1'(x_1)} \tag{7.39}$$

$$\boxed{M_2(x_1) = -H_{23}^c\bar{u}_2''(x_1) - H_{22}^c\bar{u}_3''(x_1)} \tag{7.40}$$

$$\boxed{M_3(x_1) = +H_{33}^c\bar{u}_2''(x_1) + H_{23}^c\bar{u}_3''(x_1)} \tag{7.41}$$

Note that we have also added the superscript $()^c$ to the stiffness coefficients to make it clear that now these quantities need to be evaluated using as the origin the modulus weighted centroid.

In many cases we know the moments and axial force and we are interested in finding the internal stresses and beam deflections. This requires to invert the above relations:

$$\bar{u}_1'(x_1) = \frac{1}{S}N_1(x_1) \tag{7.42}$$

$$\bar{u}_2''(x_1) = \frac{H_{23}^c}{\Delta_H}M_2(x_1) + \frac{H_{22}}{\Delta_H}M_3(x_1) \tag{7.43}$$

$$\bar{u}_3''(x_1) = -\frac{H_{33}^c}{\Delta_H}M_2(x_1) - \frac{H_{23}}{\Delta_H}M_3(x_1) \tag{7.44}$$

With $\Delta_H = H_{22}^c H_{33}^c - H_{23}^c H_{23}^c$.

The stresses can then be written as:

$$\sigma_{11} = E\left[\frac{N_1}{S} + x_3\frac{H_{33}^c M_2 + H_{23}^c M_3}{\Delta_H} - x_2\frac{H_{23}^c M_2 + H_{22}^c M_3}{\Delta_H}\right] \tag{7.45}$$

which can be rearranged in a more useful form as:

$$\sigma_{11} = E\left[\frac{N_1}{S} - \frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta_H}M_2 - \frac{x_2 H_{22}^c - x_3 H_{23}^c}{\Delta_H}M_3\right] \tag{7.46}$$

7.4.2 Equilibrium equations

The equilibrium equations for the general beam theory we are developing will be derived with the same considerations as we did in Section 7.3.2 with two modifications: 1) addition of equilibrium of moments in the $\mathbf{e}_2$ direction, 2) contribution of the axial force. Figures 7.14(a) and 7.14(b) show a free-body diagram of a beam slice subjected to both axial and transverse loads in two orthogonal but otherwise arbitrary directions (i.e, the loading direction does not necessarily match the principal axis of the cross section of the beam). The internal and external loads are shown in preparation for enforcing equilibrium.

From figure 7.14(a) we obtain the following relations for the axial N_1 and shear V_2 forces, and the bending moment M_3 in the ($\mathbf{e}_1$,$\mathbf{e}_2$) plane:

$$\begin{cases} \dfrac{dN_1}{dx_1} = -p_1(x_1) \\[2ex] \dfrac{dV_2}{dx_1} = -p_2(x_1) \\[2ex] \dfrac{dM_3}{dx_1} + V_2 = x_{2a}p_1(x_1) \end{cases} \tag{7.47}$$

From figure 7.14(b) we obtain the following relations for the axial N_1 and shear V_3 forces, and the bending moment M_2 in the ($\mathbf{e}_1$,$\mathbf{e}_3$) plane:

$$\left\{\begin{aligned} &\frac{dN_1}{dx_1} = -p_1(x_1) \\ &\frac{dV_3}{dx_1} = -p_3(x_1) \\ &\frac{dM_2}{dx_1} - V_3 = -x_{3a}p_1(x_1) \end{aligned}\right. \tag{7.48}$$

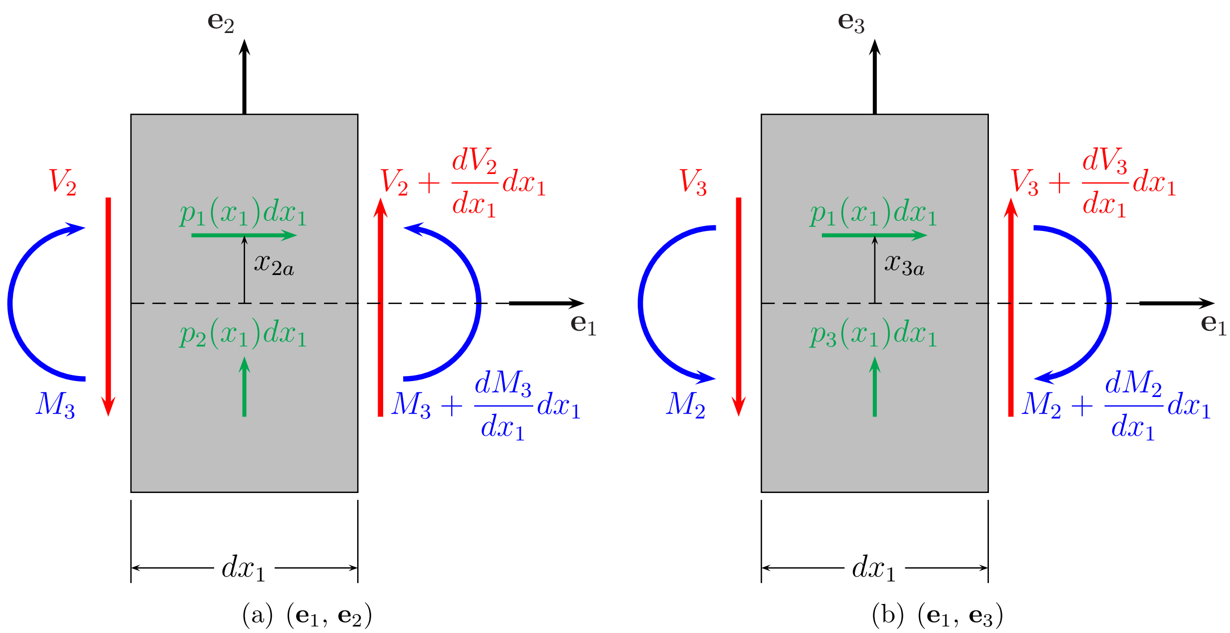

Figure 7.14: Equilibrium in both, $(\mathbf{e}_1,\mathbf{e}_2)$ and $(\mathbf{e}_1,\mathbf{e}_3)$ planes of a beam slice subjected to axial and transverse loads in general directions.

These equations can be combined by differentiating the moment equations and replacing the shear force equations in them:

$$\begin{aligned} \frac{d^2M_3}{dx_1^2} &= \frac{d}{dx_1}\left(-V_2 + x_{2a}p_1(x_1)\right) \\ &= -\frac{dV_2}{dx_1} + \frac{d}{dx_1}\left(x_{2a}p_1(x_1)\right) \\ &= p_2(x_1) + \frac{d}{dx_1}\left(x_{2a}p_1(x_1)\right) \end{aligned}$$

$$\begin{aligned} \frac{d^2M_2}{dx_1^2} &= \frac{d}{dx_1}\left(V_3 - x_{3a}p_1(x_1)\right) \\ &= \frac{dV_3}{dx_1} - \frac{d}{dx_1}\left(x_{3a}p_1(x_1)\right) \\ &= -p_3(x_1) - \frac{d}{dx_1}\left(x_{3a}p_1(x_1)\right) \end{aligned}$$

To summarize, the two equilibrium equations are:

$$\boxed{\begin{aligned} \frac{d^2 M_2}{dx_1^2} &= -p_3(x_1) - \frac{d}{dx_1}\left(x_{3a} p_1(x_1)\right) \\ \frac{d^2 M_3}{dx_1^2} &= p_2(x_1) + \frac{d}{dx_1}\left(x_{2a} p_1(x_1)\right) \end{aligned}} \tag{7.49}$$

The main peculiarity in these equations is the appearance of the terms involving the axial distributed force p_1 multiplied by the operative moment arm. This is a direct result of the fact that we cannot assume *a priori* that this force will be applied at the modulus-weighted centroid and may, thus, produce a contribution to the bending moment.

7.4.3 Governing equations

Replacing the sectional constitutive laws from Section 7.4.1 into the equations from the previous section, we obtain the governing equations:

$$\begin{cases} (S\overline{u}_1')' = -p_1 \\ \\ (H^c_{33}\overline{u}_2'' + H^c_{23}\overline{u}_3'')'' = p_2 + (x_{2a}p_1)' \\ (H^c_{23}\overline{u}_2'' + H^c_{22}\overline{u}_3'')'' = p_3 + (x_{3a}p_1)' \end{cases} \tag{7.50}$$

Concept Question 7.4.2. Observe the governing equations and try to answer the following questions:

1. What is the main difficulty in solving these equations compared to simple beam theory?
   ■ **Solution:** Clearly, the main problem in solving these equations is that they constitute a coupled system of ODEs. ■

2. Can you think of any situations in which the solution of the fourth order coupled system of ODEs can be avoided? ■ **Solution:** We can avoid solving the system when the beam problem is statically determinate. In this case we can figure out the resultant force distribution from equilibrium exclusively and we need only solve the second order equations for the sectional constitutive laws in order to figure out the stresses and the deflections. ■

Boundary conditions When the system has to be solved, appropriate boundary conditions must be provided. Depending on the type of idealization of the physical system, type of support and loading, we can have a combination of imposed displacements, constrained rotations, forces or moments, i.e.

$$\overline{u}_1 = \overline{u}_2 = \overline{u}_3 = 0 \qquad \text{and} \qquad \overline{u}_2' = \overline{u}_3' = 0 \tag{7.51}$$

$$\begin{cases} N_1 = P_1 \\ V_2 = P_2 \ , \ V_3 = P_3 \\ M_3 = -x_{2a} P_1 \ , \ M_2 = x_{3a} P_1 \end{cases} \tag{7.52}$$

These can be written as a function of derivatives of the beam deflections. $\overline{u}_1$, $\overline{u}_2$ and $\overline{u}_3$:

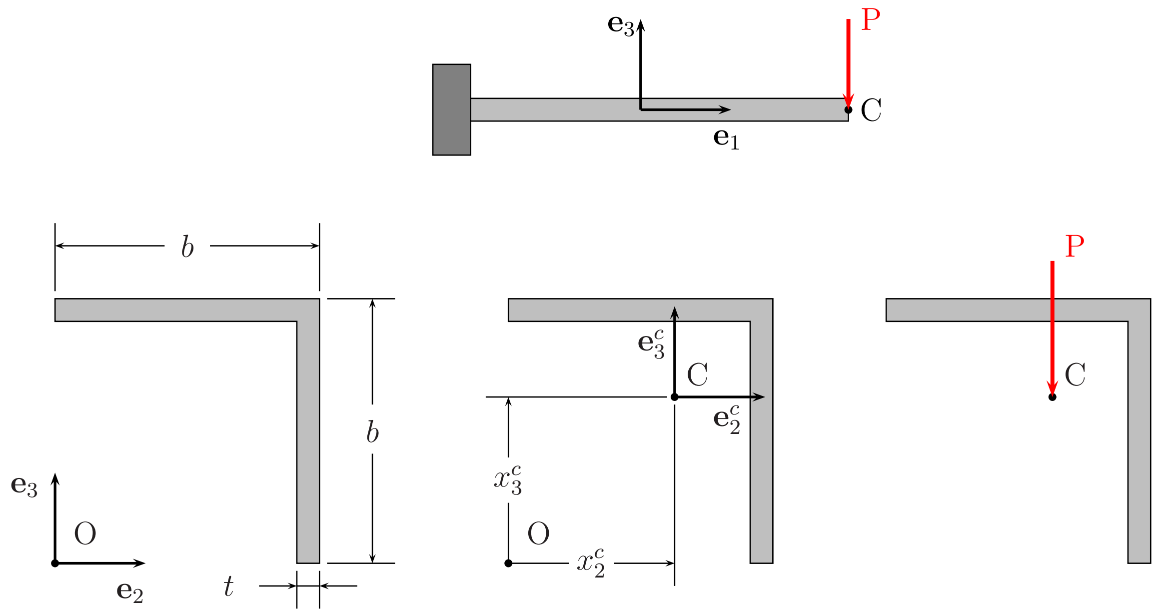

Figure 7.15: Cantilever beam with a L-shaped cross section.

Concept Question 7.4.3. *bending of a beam with a L-shaped cross section.* Let us consider a cantilever beam with a L-shaped cross section as depicted in Figure 7.15. It is assumed that the beam is made of a linear homogeneous material, in this context fully described by its Young's modulus $E = 2 \times 10^{11}$ GPa. The cross section of the beam is 0.1 m wide and high (b); its thickness, t, is equal to 2 mm, and, its length, l, is equal to 2 m. A load P of 200 N is applied at the free-end of the beam, more precisely at C, it modulus-weighted centroid.

1. Compute the coordinates (x_2^c, x_3^c) of the modulus weighted centroid of the section with respect to the origin O. ■ **Solution:** We use (7.38) for which we need to compute the axial stiffness (S) of the cross section:

$$S = \underbrace{Et(b-t)}_{\mathbf{e}_2 \text{ beam}} + \underbrace{Etb}_{\mathbf{e}_3 \text{ beam}} \approx 2Etb$$

as well the first moments of area w.r.t to $\mathbf{e}_3$ (S_2) and $\mathbf{e}_2$ (S_3) which are equal due to the symmetry of the cross section:

$$\begin{aligned} S_2 &= \int_A E x_2 dA = \underbrace{Et \int_0^{b-t} x_2 dx_2}_{\mathbf{e}_2 \text{ beam}} + \underbrace{Eb \int_{b-t}^{b} x_2 dx_2}_{\mathbf{e}_3 \text{ beam}} \\ &= \frac{Et}{2}(b-t)^2 + \frac{Eb}{2}\left(b^2 - (b-t)^2\right) \approx \frac{Etb^2}{2} + Etb^2 = \frac{3Etb^2}{2} \end{aligned}$$

Hence, the coordinates of the centroid are:

$$x_2^c = \frac{S_2}{S} = \frac{\frac{3Etb^2}{2}}{2Etb} = \frac{3b}{4} = x_3^c$$

■

2. Compute the bending stiffnesses in the coordinate system (x_2^c, x_3^c). ■ **Solution:** We use the relations given in Table 7.1:

$$\begin{aligned} H_{33}^c &= \int_A E(x_2 - x_2^c)^2 dA = \underbrace{Et \int_0^{b-t} (x_2 - x_2^c)^2 dx_2}_{\mathbf{e}_2 \text{ beam}} + \underbrace{Eb \int_{b-t}^{b} (x_2 - x_2^c)^2 dx_2}_{\mathbf{e}_3 \text{ beam}} \\ &= Et \int_{-\frac{3b}{4}}^{\frac{b}{4}-t} x_2^2 dx_2 + Eb \int_{\frac{b}{4}-t}^{\frac{b}{4}} x_2^2 dx_2 \\ &= \frac{Et}{3}\left(\left(\frac{b}{4}-t\right)^3 + \left(\frac{3b}{4}\right)^3\right) + \frac{Eb}{3}\left(\left(\frac{b}{4}\right)^3 - \left(\frac{b}{4}-t\right)^3\right) \\ &\approx \frac{Et}{3}\left(\frac{b^3}{64} + \frac{27b^3}{64}\right) + \frac{Eb}{3}\left(\frac{b^3}{64} - \frac{b^3}{64} + \frac{3tb^2}{16}\right) = Etb^3\left(\frac{7}{48} + \frac{3}{48}\right) = \frac{5}{24}Etb^3 \\ H_{33}^c &= H_{22}^c = \frac{5}{24}Etb^3 \\ H_{23}^c &= \int_A E(x_2 - x_2^c)(x_3 - x_3^c) dA \\ &= \underbrace{E \int_{-\frac{3b}{4}}^{\frac{b}{4}-t} x_2 dx_2 \int_{\frac{b}{4}-t}^{\frac{b}{4}} x_3 dx_3}_{\mathbf{e}_2 \text{ beam}} + \underbrace{E \int_{\frac{b}{4}-t}^{\frac{b}{4}-t} x_2 dx_2 \int_{-\frac{3b}{4}}^{\frac{b}{4}} x_3 dx_3}_{\mathbf{e}_3 \text{ beam}} \\ &= \frac{E}{4} \underbrace{\left(\left(\frac{b}{4}-t\right)^2 - \frac{9b^2}{16}\right)}_{-\frac{b^2}{2}} \underbrace{\left(\frac{b^2}{16} - \left(\frac{b}{4}-t\right)^2\right)}_{\frac{tb}{2}} + \frac{E}{4} \underbrace{\left(\frac{b^2}{16} - \left(\frac{b}{4}-t\right)^2\right)}_{\frac{tb}{2}} \underbrace{\left(\frac{b^2}{16} - \frac{9b^2}{16}\right)}_{-\frac{b^2}{2}} \\ H_{23}^c &= -\frac{Etb^3}{8} \end{aligned}$$

$$\Delta_H = H_{22}^c H_{33}^c - H_{23}^c H_{23}^c = \frac{(Etb^3)^2}{36}$$

■

3. Compute the maximum tensile and compressive stresses in the L-shaped cross section.
   ■ **Solution:** The maximum tensile and compressive stresses in the cross section which at the clamped end of the cantilever beam because it is where the moment M_2 is maximum and equal to Pl. Because the load P at the free end of the cantilever beam is applied at the modulus weighted centroid, the other resultants, M_3 and N_1 are null. Thus, making use of (7.46), σ_{11} is equal to:

$$\sigma_{11} = -E\frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta_H}\mathrm{P}l$$

   The maximum traction stress is found at point $(\frac{b}{4}, \frac{b}{4})$ and the maximum compressive stress is found at point $(\frac{b}{4}, -\frac{3b}{4})$, both at the clamped end of the beam.

$${}^t\sigma_{11} = EPl\frac{\frac{b}{4}H_{33} - \frac{b}{4}H_{23}}{\Delta_H} = \frac{3Pl}{h^2 t}.$$

$${}^c\sigma_{11} = -EPl\frac{\frac{b}{4}H_{33} + \frac{3b}{4}H_{22}}{\Delta_H} = -\frac{9Pl}{2h^2 t}.$$

   ■

4. Determine the neutral axis orientation with respect to $\mathbf{e}_2$ ■ **Solution:** The neutral axes can be defined as the axes for which the stress $\sigma_{11} = 0$:

$$-E\frac{x_2 H_{23}^c - x_3 H_{33}^c}{\Delta_H}\mathrm{P}l = 0$$

   The previous equation defines a line in the $(\mathrm{C}, \mathbf{e}_2^c, \mathbf{e}_3^c)$ defined by the equation: $x_2 H_{23}^c - x_3 H_{33}^c = 0$. Its orientation is thus given by the angle β defined as:

$$\tan(\beta) = \frac{H_{23}}{H_{33}} \Rightarrow \beta = -\mathrm{atan}\left(\frac{3}{5}\right) = -30.964^\circ$$

   ■

7.4.4 Decoupling the problem

In section 7.4.1 we wrote both, the axial force N_1 and the bending moments M_2 and M_3 as a function of the axial and bending sectional stiffnesses $S, S_2, S_3, H_{22}, H_{33}, H_{23}$. These relations were simplified if we referred all our coordinates to the modulus-weighted centroid of the cross section, in which case $S_2 = 0$ and $S_3 = 0$). From the equations of equilibrium obtained in section 7.4.2 we obtain the following matrix system:

$$\left\{\begin{array}{c} N_1(x_1) \\ M_2(x_1) \\ M_3(x_1) \end{array}\right\} = \begin{bmatrix} S & 0 & 0 \\ 0 & -H_{22}^c & -H_{23}^c \\ 0 & H_{23}^c & H_{33}^c \end{bmatrix} \left\{\begin{array}{c} \overline{u}_1'(x_1) \\ \overline{u}_3''(x_1) \\ \overline{u}_2''(x_1) \end{array}\right\} \tag{7.53}$$

Here, we have a partially uncoupled problem. Indeed, the axial force is only related to the first derivative of the displacement along the $\mathbf{e}_1$ direction but the displacement components u_2 and u_3 are coupled because of the presence of the non-zero cross bending stiffness H_{23}. In order to solve the partially uncoupled problem, the main idea is to determine the directions that the axis of the beam should match in order to the problem to be fully uncoupled. In other words, we want the matrix in equation 7.53 to be diagonal, without any coupling term which leads to:

$$H_{23}^c = \int_{A(x_1)} E x_2 x_3 dA = 0 \tag{7.54}$$

which also defines the *principal centroidal axes of bending.* For that purpose, we determine the reference frame (denoted with a $*$ in the following) where the matrix is diagonal, also well-known as principal directions, and define the components of the diagonal matrix which are the principal/eigen values. Of note, the obtained diagonal matrix will satisfy the two equilibrium relations in equation 7.49.

Concept Question 7.4.4. *Decoupled constitutive laws.* Let's consider the associated fully decoupled problem where the matrix in equation 7.53 is diagonal and written as a function of S^*, H_{22}^{c*} and H_{33}^{c*} as follows:

$$\begin{bmatrix} S^* & 0 & 0 \\ 0 & -H_{22}^{c*} & 0 \\ 0 & 0 & H_{33}^{c*} \end{bmatrix} \tag{7.55}$$

Write the constitutive laws (expression of axial stress distribution σ_{11}) for this fully decoupled problem as a function of S^*, H_{22}^{c*} and H_{33}^{c*}. ■ **Solution:**

$$\overline{u}_1'^* = \frac{N_1^*}{S^*} \quad , \quad \overline{u}_2''^* = \frac{M_3^*}{H_{33}^{c*}} \quad , \quad \overline{u}_3''^* = -\frac{M_2^*}{H_{22}^{c*}}$$

hence, the corresponding axial stress distribution reads:

$$\sigma_{11}^* = E\left(\frac{N_1^*}{S^*} + x_3^*\frac{M_2^*}{H_{22}^{c*}} - x_2^*\frac{M_3^*}{H_{33}^{c*}}\right)$$

■

Concept Question 7.4.5. *Decoupled governing equations.* For the same fully decoupled problem as above, write the three governing equations as a function of S^*, H_{22}^{c*} and H_{33}^{c*}. ■
Solution: For the axial force, the first relation of equation 7.50 is written as follows:

$$\frac{d}{dx_1^*}\left(S^*\overline{u}_1'\right) = -p_1^*$$

For the bending moment M_2, the second relation of equation 7.50 is written as follows:

$$\frac{d^2}{dx_1^{*2}}\left(H_{33}^{c*}\overline{u}_2''^*\right) = p_2^* + \frac{d}{dx_1^*}\left(x_{2a}^* p_1^*\right)$$

For the bending moment M_3, the third relation of equation 7.50 is written as follows:

$$\frac{d^2}{dx_1^{*2}}\left(H_{22}^{c*}\overline{u}_3^{\prime\prime *}\right) = p_3^* + \frac{d}{dx_1^*}\left(x_{3a}^* p_1^*\right)$$

■

Concept Question 7.4.6. *Principal centroidal axes of bending.* We consider the fully decoupled problem associated with the diagonal matrix in equation 7.55. Herein, the centroidal axes of bending, also defined as the reference frame, correspond to the principal direction of the diagonal matrix. In this exercice we want to determine both the principal directions and the eigen values S^*, H_{22}^{c*} and H_{33}^{c*}.

Show that $S^* = S$ and that leads to diagonalize a 2×2 matrix you specify.

■ **Solution:** Herein, the non-diagonal 3×3 matrix in equation 7.53 is partially diagonal and S is an eigen value, hence $S^* = S$. This matrix can be diagonalized by diagonalizing the following 2×2 sub-matrix:

$$\begin{bmatrix} -H_{22}^c & -H_{23}^c \\ H_{23}^c & H_{33}^c \end{bmatrix}$$

■

Using either the general formulae of the diagonalization of a 2×2 matrix or the Mohr's circle relations, to define an expression for both the eigen values and the principal directions.

■ **Solution:** By definition, any point on the centroidal axis of bending are such that $H_{23}^c = 0$. The orientation (α) of the centroidal axes of bending is written as follows:

$$\tan(2\alpha) = \frac{2H_{23}^c}{H_{33}^c - H_{22}^c}$$

and the two eigen values reads:

$$H_{22}^{c*} = \frac{H_{33}^c + H_{22}^c}{2} - \Delta \quad , \quad H_{33}^{c*} = \frac{H_{33}^c + H_{22}^c}{2} + \Delta$$

■

To summarize, solving a three-dimensional general beam problem consists in decoupling the problem in three separate problems by expressing the compatibility equations, the constitutive laws and the governing equations in the reference frame characterized by the principal centroidal axes of bending. For that purpose, we follow the steps listed below:

- (i) Compute the centroid of the section using the equation 7.38
- (ii) Compute the bending stiffnesses in this axis system using the relations in the table 7.1
- (iii) Compute the orientation of the principal axes of bending using the equation 7.4.6
- (iv) Compute the principal bending stiffnesses using equation 7.4.6