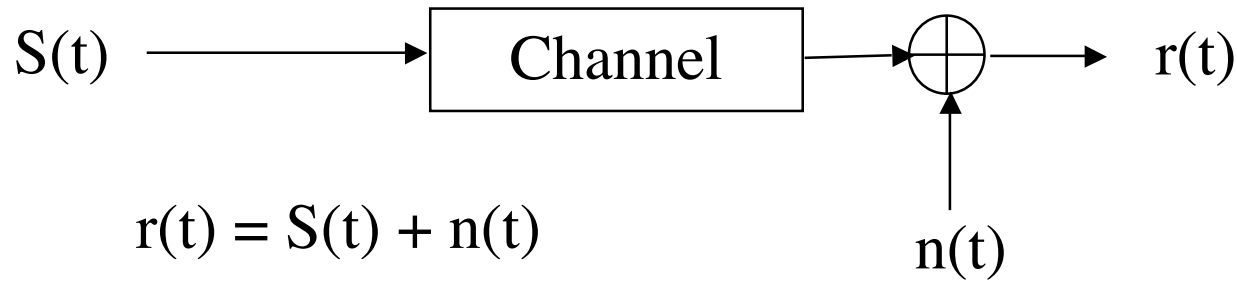

Lectures 8 & 9: Signal Detection: matched filter

Eytan Modiano

AA Dept.

Noise in communication systems



- **Noise is additional “unwanted” signal that interferes with the transmitted signal**
 - Generated by electronic devices
- **The noise is a random process**
 - Each “sample” of $n(t)$ is a random variable
- **Typically, the noise process is modeled as “Additive White Gaussian Noise” (AWGN)**
 - White: Flat frequency spectrum
 - Gaussian: noise distribution

Random Processes

- The auto-correlation of a random process $x(t)$ is defined as
 - $R_{xx}(t_1, t_2) = E[x(t_1)x(t_2)]$
- A random process is Wide-sense-stationary (WSS) if its mean and auto-correlation are not a function of time. That is
 - $m_x(t) = E[x(t)] = m$
 - $R_{xx}(t_1, t_2) = R_x(\tau)$, where $\tau = t_1 - t_2$
- If $x(t)$ is WSS then:
 - $R_x(\tau) = R_x(-\tau)$
 - $|R_x(\tau)| \leq |R_x(0)|$ (max is achieved at $\tau = 0$)
- The power content of a WSS process is:

$$P_x = E\left[\lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t) dt\right] = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} R_x(0) dt = R_x(0)$$

Power Spectrum of a random process

- If $x(t)$ is WSS then the power spectral density function is given by:

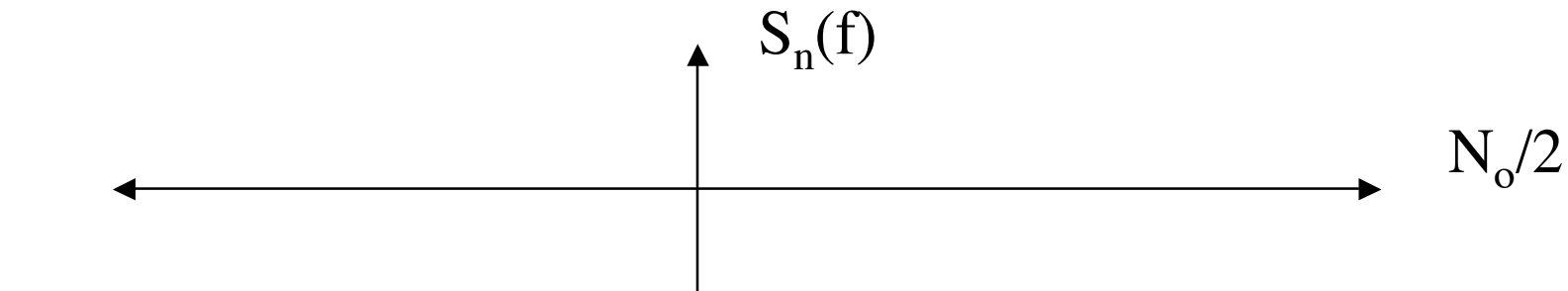
$$S_x(f) = F[R_x(\tau)]$$

- The total power in the process is also given by:

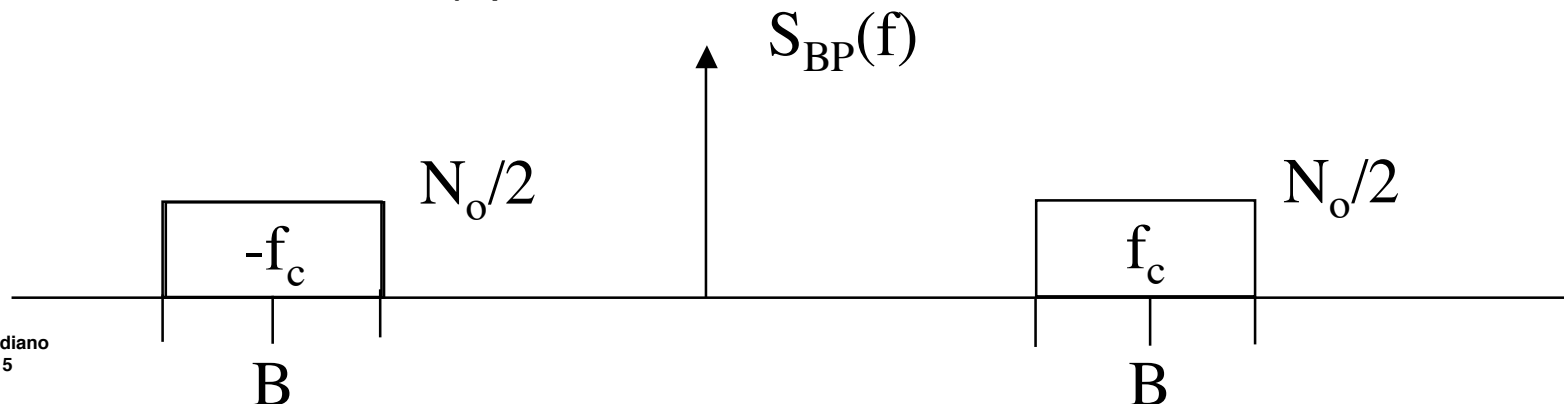
$$\begin{aligned} P_x &= \int_{-\infty}^{\infty} S_x(f) df = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} R_x(t) e^{-j2\pi ft} dt \right] df \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} R_x(t) e^{-j2\pi ft} df \right] dt \\ &= \int_{-\infty}^{\infty} R_x(t) \left[\int_{-\infty}^{\infty} e^{-j2\pi ft} df \right] dt = \int_{-\infty}^{\infty} R_x(t) \delta(t) dt = R_x(0) \end{aligned}$$

White noise

- The noise spectrum is flat over all relevant frequencies
 - White light contains all frequencies



- Notice that the total power over the entire frequency range is infinite
 - But in practice we only care about the noise content within the signal bandwidth, as the rest can be filtered out
- After filtering the only remaining noise power is that contained within the filter bandwidth (B)



AWGN

- The effective noise content of bandpass noise is BN_o
 - Experimental measurements show that the pdf of the noise samples can be modeled as zero mean gaussian random variable

$$f_x(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2 / 2\sigma^2}$$

- AKA Normal r.v., $N(0, \sigma^2)$
 - $\sigma^2 = P_x = BN_o$
- The CDF of a Gaussian R.V.,

$$F_x(\alpha) = P[X \leq \alpha] = \int_{-\infty}^{\alpha} f_x(x) dx = \int_{-\infty}^{\alpha} \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2 / 2\sigma^2} dx$$

- This integral requires numerical evaluation
 - Available in tables

AWGN, continued

- $X(t) \sim N(0, \sigma^2)$
- $X(t_1), X(t_2)$ are independent unless $t_1 = t_2$

-

$$R_x(\tau) = E[X(t + \tau)X(t)] = \begin{cases} E[X(t + \tau)]E[X(t)] & \tau \neq 0 \\ E[X^2(t)] & \tau = 0 \end{cases}$$

$$= \begin{cases} 0 & \tau \neq 0 \\ \sigma^2 & \tau = 0 \end{cases}$$

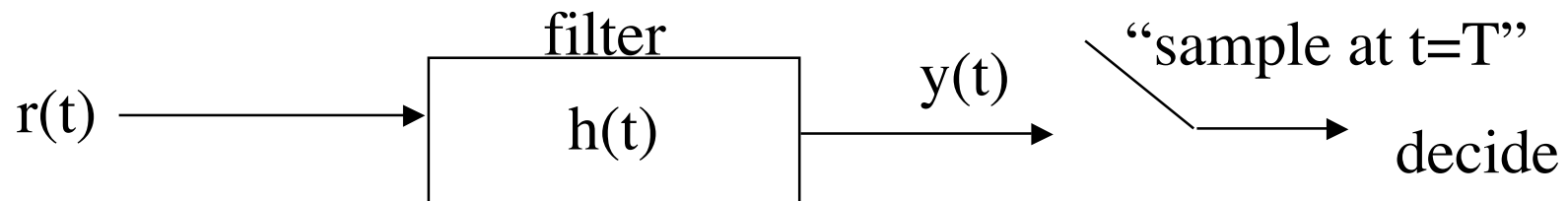
- $R_x(0) = \sigma^2 = P_x = BN_o$

Detection of signals in AWGN

Observe: $r(t) = S(t) + n(t)$, $t \in [0, T]$

Decide which of $S_1, \dots, S_m$ was sent

- **Receiver filter**
 - Designed to maximize signal-to-noise power ratio (SNR)



- **Goal: find $h(t)$ that maximized SNR**

Receiver filter

$$y(t) = r(t) * h(t) = \int_0^t r(\tau) h(t - \tau) d\tau$$

$$\text{Sampling at } t = T \Rightarrow y(T) = \int_0^T r(\tau) h(T - \tau) d\tau$$

$$r(\tau) = s(\tau) + n(\tau) \Rightarrow$$

$$y(T) = \int_0^T s(\tau) h(T - \tau) d\tau + \int_0^T n(\tau) h(T - \tau) d\tau = Y_s(T) + Y_n(T)$$

$$SNR = \frac{Y_s^2(T)}{E[Y_n^2(T)]} = \frac{\left[\int_0^T s(\tau) h(T - \tau) d\tau \right]^2}{\frac{N_0}{2} \int_0^T h^2(T - t) dt} = \frac{\left[\int_0^T h(\tau) s(T - \tau) d\tau \right]^2}{\frac{N_0}{2} \int_0^T h^2(T - t) dt}$$

Matched filter: maximizes SNR

Cauchy - Schwartz Inequality :

$$\left[\int_{-\infty}^{\infty} g_1(t) g_2(t) dt \right]^2 \leq \int_{-\infty}^{\infty} (g_1(t))^2 dt \int_{-\infty}^{\infty} (g_2(t))^2 dt$$

Above holds with equality iff: $g_1(t) = c g_2(t)$ for arbitrary constant c

$$SNR = \frac{\left[\int_0^T s(\tau) h(T - \tau) d\tau \right]^2}{\frac{N_0}{2} \int_0^T h^2(T - t) dt} \leq \frac{\int_0^T (s(\tau))^2 d\tau \int_0^T h^2(T - \tau) d\tau}{\frac{N_0}{2} \int_0^T h^2(T - t) dt} = \frac{2}{N_0} \int_0^T (s(\tau))^2 d\tau = \frac{2E_s}{N_0}$$

Above maximum is obtained iff: $h(T - \tau) = c S(\tau)$

$$\Rightarrow h(t) = c S(T - t) = S(T - t)$$

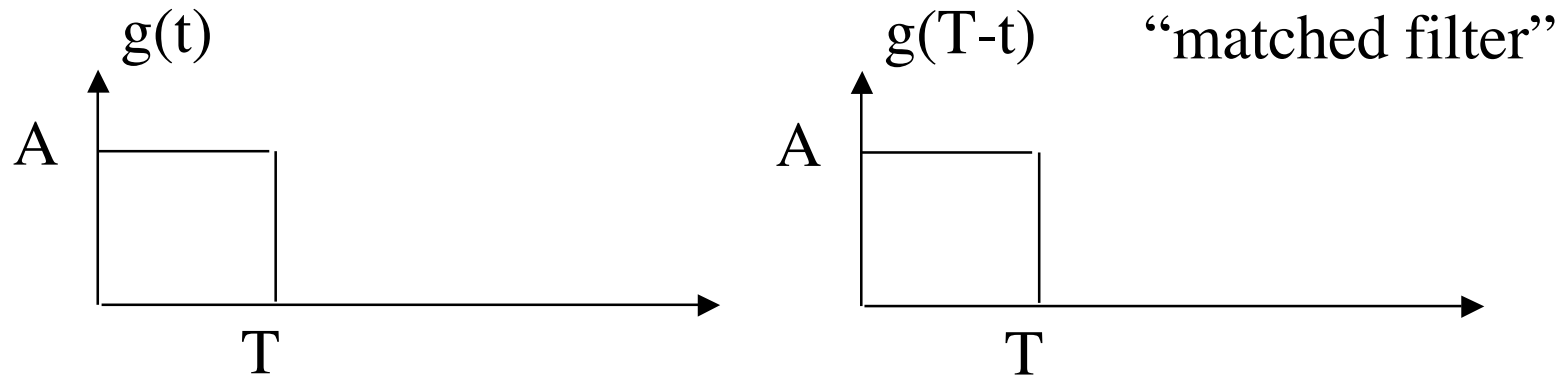
$h(t)$ is said to be “matched” to the signal $S(t)$

Example: PAM

$$S_m(t) = A_m g(t), \quad t \in [0, T]$$

A_m is a constant: Binary PAM $A_m \in \{0, 1\}$

Matched filter is matched to $g(t)$



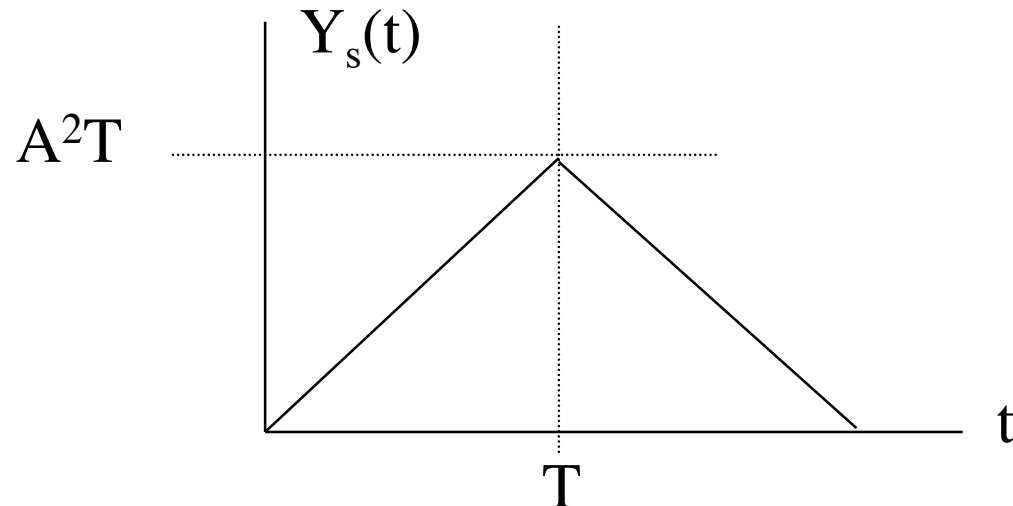
Example, continued

$$Y_s(t) = \int_0^t S(\tau)h(t - \tau)d\tau, \quad h(t) = g(T - t) \Rightarrow h(t - \tau) = g(T + \tau - t)$$

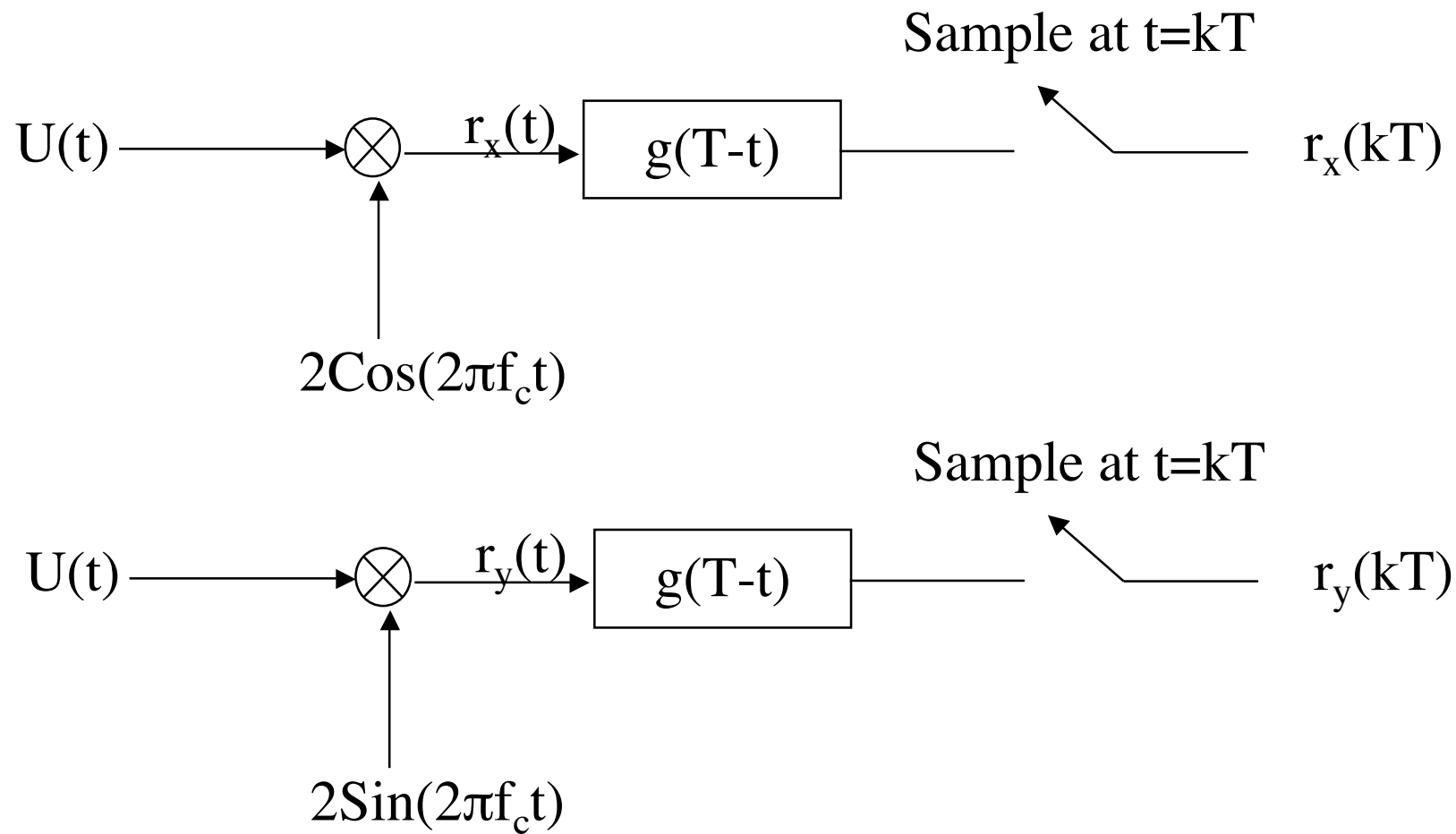
$$Y_s(t) = \int_0^t g(\tau)g(T + \tau - t)d\tau = \int_0^t g(\tau)g(T - t + \tau)d\tau$$

$$Y_s(T) = \int_0^T g^2(\tau)d\tau$$

- **Sample at $t=T$ to obtain maximum value**



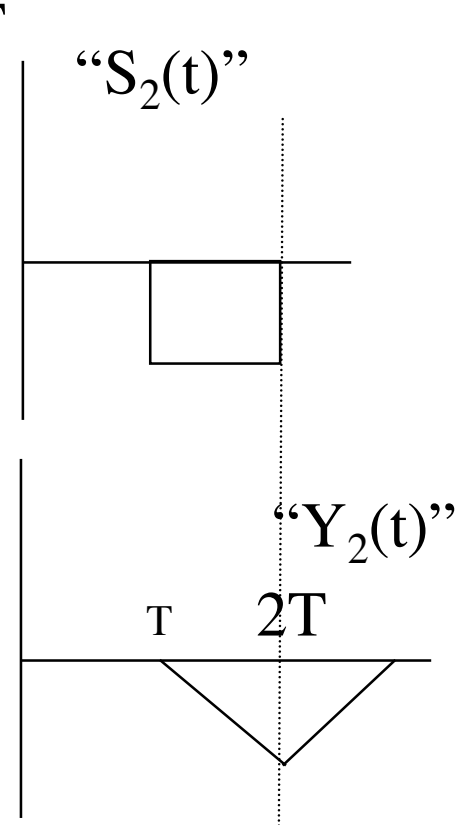
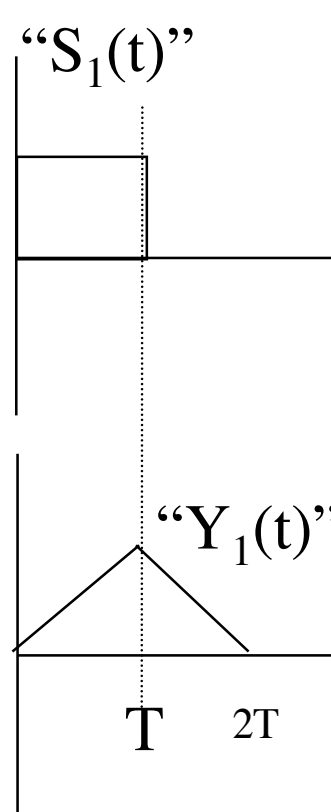
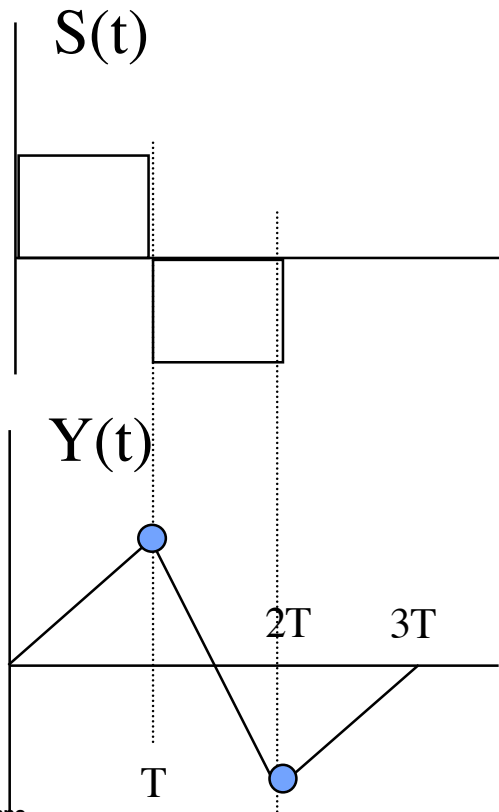
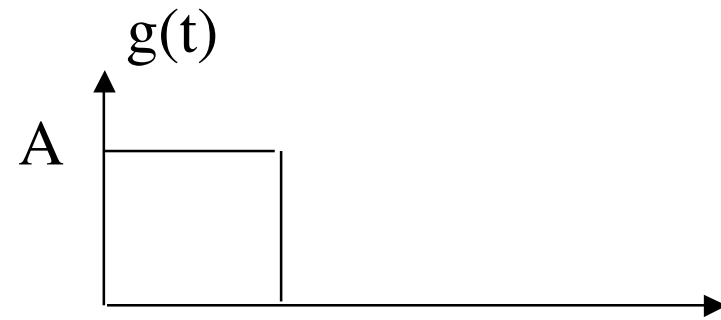
Matched filter receiver



Binary PAM example, continued

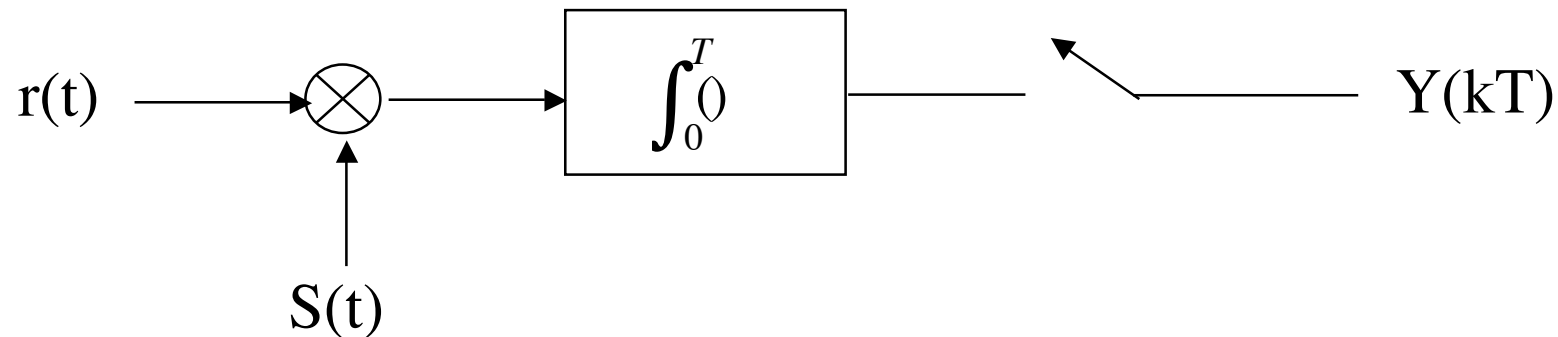
$$0 \Rightarrow S_1 = g(t)$$

$$1 \Rightarrow S_2 = -g(t)$$



Alternative implementation: correlator receiver

$$r(t) = S(t) + n(t)$$



$$Y(T) = \int_0^T r(t)S(t) = \int_0^T S^2(t) + \int_0^T n(t)S(t) = Y_s(T) + Y_n(T)$$

Notice resemblance to matched filter