Massachusetts Institute of Technology

16.410-13 Principles of Autonomy and Decision Making

Due: Monday, 11/22/04

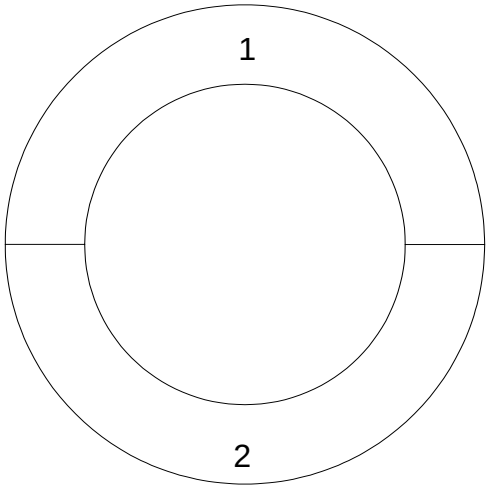
Paper solutions are due no later than 5pm on Monday, 11/22/04. Please give solutions to the course secretary, Brian O' Conaill, at his desk outside of 33-330.

Objectives

The purpose of this problem set is to develop a grounded understanding of Solutions to Markov Decision Processes and Mixed Integer Programs.

Problem 1 – Markov Decision Processes

Consider the circular racetrack shown below.



It has two sections: section 1 is dry, and section 2 is slippery. A race car on this track will crash if it is on section 2 and is going too fast.

Suppose we want to give the race car driver advice about how fast to go in each section. We will model this system using a simple deterministic MDP, where the **state** is simply the section number that the race car is in, and the **action** is one of three speeds: 0, 20, and 40 mph.

The transition function is as follows:

s	a	s+1
1	0	1
1	20	2
1	40	2
2	0	2
2	20	1
2	40	2

The reward function is

s	a	R
1	0	0
1	20	15
1	40	45
2	0	0
2	20	10
2	40	0

Note that if the driver goes 40 mph in section 2, he will stay in section 2, and his reward will be 0 (he will crash).

Assume that the discount rate γ is 0.8

Part A. What is the value function and optimal policy for a 2-step horizon?

Solution:

$V^*_1(1) = 45$ $\pi^*_1(1) = 40$

$V^*_1(2) = 10$ $\pi^*_1(2) = 20$

$V^*_2(1) = 45 + 0.8 * 10 = 53$ $\pi^*_2(1) = 40$

$V^*_2(2) = 10 + 0.8 * 45 = 46$ $\pi^*_2(2) = 20$

Part B. How do these change if the transition function is altered to the table below? Give the value function and optimal policy, and comment on the reason for the changes from Part A.

s	a	s+1
1	0	1
1	20	2
1	40	2
2	0	2
2	20	2
2	40	1

Solution:

The difference from part A is that there is now a reward for staying in state 2, but no reward for transitioning from 2 to 1. For the two step horizon the solution changes to:

$V^*_1(1) = 45$ $\quad\quad$ $\pi^*_1(1) = 40$

$V^*_1(2) = 10$ $\quad\quad$ $\pi^*_1(2) = 20$

$V^*_2(1) = 45 + 0.8 * 10 = 53$ $\quad\quad$ $\pi^*_2(1) = 40$

$V^*_2(2) = 0 + 0.8 * 45 = 36$ $\quad\quad$ $\pi^*_2(2) = 20$

Here the value of V(2) decreases; however, the policy doesn't change. This is because the reward for going from 1 to 2 at 40 mph (45) is still more than twice the reward of staying at 2 at 20 mph (10). For the policy to change the reward of staying at 2 would need to go above 25.

Problem 2 Integer Programming and Branch and Bound

Part A Formulation Using Integer Programming

The Transportation Security Administration needs to maintain an all-night security gate at a busy international airport. All its employees work in **eight hour shifts**. The number of employees needed to run the gate varies according to time of day, because fewer or more passengers travel during those times. From 12 midnight-4am, 25 people are needed; from 4am-8am, 45 people; from 8am-12noon, 85 people, from 12noon-4pm, 120; from 4pm-8pm, 55 people; from 8pm-12midnight, 33 people.

Part A.1

Write an integer program whose solution gives the minimum-employee solution to the TSA's staffing problem. Include a description in words of what your variables mean. Explain any key modeling decisions in your encoding.

Solution:

xi = number of people who start their shift at 4*i hours after midnight.

Minimize $z = x_0 + x1 + x2 + x3 + x4 + x5$

Subject to
$x_0 + x_5 \geq 25;$
$x_0 + x_1 \geq 45;$
$x_1 + x_2 \geq 85;$
$x_2 + x_3 \geq 120;$
$x_3 + x_4 \geq 55;$
$x_4 + x_5 \geq 33;$
$xi \geq 0$, integer(xi);

Part A.2

Appearances are important, and the TSA is getting complaints because of the surplus employees that are hanging around Dunkin' Donuts during their shifts. Modify your integer program to minimize the maximum number of unnecessary employees in any one shift. Give your integer program and an explanation for any modifications that you made from part A.1.

Solution:

As before,

xi = number of people who start their shift at 4*i hours after midnight.

We make z to be the max of the surplus employees by adding constraints that make z an upper bound on this surplus, and then minimize z, to make it a least upper bound.

Minimize z

Subject to
$z \geq x_0 + x_5 - 25;$
$z \geq x_1 + x_2 - 45;$
$z \geq x_2 + x_3 - 85;$
$z \geq x_3 + x_4 - 120;$
$z \geq x_4 + x_5 - 55;$
$z \geq x_5 + x_6 - 33;$

$x_0 + x_5 \geq 25;$
$x_0 + x_1 \geq 45;$
$x_1 + x_2 \geq 85;$
$x_2 + x_3 \geq 120;$
$x_3 + x_4 \geq 55;$
$x_4 + x_5 \geq 33;$
$xi \geq 0$, integer(xi);

Part B Solving Integer Programs using Branch and Bound

Solve the following mixed integer linear program using Branch and Bound.

Minimize $z = 8x_1 + 3x_2 + 18b_1 + 20b_2$

subject to
$x_1 + x_2 \geq 8$
$x_1 \leq 15b_1$
$x_2 \leq 10b_2$

$b_1 + b_2 \leq 1$
b_1, b_2 are binary (i.e., $b_i \in \{1,0\}$).

Part B.1 Branch and Bound Search Tree

Construct a branch and bound search tree that augments the template tree given below. Branch on the binary variables in the **following order**: b_1, b_2. Evaluate the **0** branch **before** the **1** branch. Cross off each node that is infeasible or fathomed. For feasible, non-fathomed nodes, give the relaxed solution and the value of Z. For fathomed nodes, give the solution and value of Z.

Part B.2 Minimum Feasible Solution

List your solution, which is the minimum feasible state:

$Z = \underline{34}\_\_$, $x_1 = \underline{-2}\_\_$, $x_2 = \underline{10}\_\_$, $b_1 = \underline{0}\_\_\_$, $b_2 = \underline{1}\_\_\_$,

Derivation of Solution:
You can solve each relaxed problem using the simplex method. This problem is simple enough that you can also solve it by inspection. The solution below is based on the use of monotonicity arguments to determine which particular constraints are active (this is called monotonicity analysis, or activity analysis). A constraint is active if the solution satisfies the constraint as an equality, rather than an inequality.

We start by initializing the incumbent to:

z*= infinity
b1=?, b2=?, x1=?, x2=?,

We then relax the binary variables at the Root Node and solve:

Root Node:

Minimize $z = 8x_1 + 3x_2 + 18b_1 + 20b_2$
subject to $x_1 + x_2 \geq 8$
$x_1 \leq 15b_1$
$x_2 \leq 10b_2$
$b_1 + b_2 \leq 1$
$b_i \in [1,0]$.

First, z monotonically decreases as x1 and x2 monotonically decrease, hence z is a minimum when

$x_1 + x_2 \geq 8 \quad \rightarrow x_1 + x_2 = 8$ (i.e., the constraint is said to be active)

Note that this argument holds unless x1 and x2 reach another constraint boundary first. We can test this at the end by checking feasibility of the solution.

Solving for x_2 produces:

$x_2 = 8 - x_1$

Substituting for x_2 into the problem and simplifying produces:

Minimize $z = 5x_1 + 18b_1 + 20b_2 + 24$
subject to $x_1 \leq 15b_1$
$8 \leq 10b_2 + x_1$
$b_1 + b_2 \leq 1$
$b_i \in [1,0]$.

Second, z monotonically decreases as x_1 and b_2 monotonically decrease, hence z is a minimum when:

$8 \leq 10b_2 + x_1 \rightarrow 8 = 10b_2 + x_1$ (i.e., active constraint).

Solving for x_1:

$x_1 = 8 - 10b_2$,

substituting for x_1 into the problem and simplifying produces:

Minimize $z = 18b_1 - 30b_2 + 64$
subject to $8 \leq 15b_1 + 10b_2$
$b_1 + b_2 \leq 1$
$b_i \in [1,0]$.

Third, z monotonically decreases as b_1 monotonically decreases, hence z is a minimum when either

$0 \leq b_1$, or $8 \leq 15b_1 + 10b_2$ is active. Assuming the former:

$b_1 = 0$ (i.e., active constraint).

Substituting for b_1 into the problem and simplifying produces:

Minimize $z = - 30b_2 + 64$
subject to $8 \leq 10b_2$

$b_2 \leq 1,\ b_i \in [1,0]$.

Finally, z monotonically decreases as b_2 monotonically **increases**, hence z is a minimum when
$b_2 \leq 1 \rightarrow b_2 = 1$

Substituting for b_2 into the equations for z, x_1, x_2 and b_2 produces:
$z = 34$
$x_1 = 8 - 10b_2 = -2$
$x_2 = 8 - x_1 = 10$

To summarize, the relaxed solution is:
$b_1 = 0, b_2 = 1, x_1 = -2, x_2 = 10, z = 34$

This relaxed solution is feasible (substitute into the inequality constraints to confirm). The integer variables take on integer values for the relaxed solution, hence this is the optimal solution for the integer program. No other search nodes need to be explored.

For those of you who missed finding the solution on the root node, below is what you would find when going to the next level. This also serves to better demonstrate branch and bound. Note, however, that going beyond the root isn't needed:

Node $b_1 = 0$ (not needed):

To solve the relaxed problem:

Minimize	$z = 8x_1 + 3x_2 + 20b_2$
subject to	$x_1 + x_2 \geq 8$
	$x_1 \leq 0$
	$x_2 \leq 10b_2$
	$b_2 \in [1,0]$.

We go through a similar process to above. First, z monotonically decreases with x1 and x2, thus:
$x_1 + x_2 \geq 8 \quad \rightarrow x_1 + x_2 = 8$ (i.e., the constraint is said to be active)
$x_2 = 8 - x_1$

Once again, remember that this type of argument only holds if x_1 and x_2 **do not** hit another constraint boundary first.

Substituting for x_2 and simplifying produces:

Minimize	$z = 5x_1 + 20b_2 + 24$
subject to	$x_1 \leq 0$
	$8 \leq 10b_2 + x_1$
	$b_2 \in [1,0]$.

Second, z monotonically decreases with x_1 and b_2, hence z is a minimum when:
$8 \leq 10b_2 + x_1 \rightarrow 8 = 10b_2 + x_1$ (i.e., active constraint).:
Solving and substituting for x_1 into the problem produces:

$x_1 = 8 - 10b_2$,

Minimize $z = -30b_2 + 64$
subject to $8 \leq 10b_2$
$b_2 \in [1,0]$.

Third, z monotonically decreases as b_1 monotonically decreases, hence z is a minimum when either
$0 \leq b_1$, or $8 \leq 15b_1 + 10b_2$ is active. Assuming the former:
$b_1 = 0$ (i.e., active constraint).

Substituting for b_1 into the problem and simplifying produces:
Minimize $z = -30b_2 + 64$
subject to $8 \leq 10b_2$
$b_2 \leq 1$, $b_i \in [1,0]$.

Finally, z monotonically decreases as b_2 monotonically **increases**, hence z is a minimum when
$b_2 \leq 1 \rightarrow b_2 = 1$

Substituting for b_2 into the equations for z, x_1, x_2 and b_2 produces:
$z = 34$
$x_1 = 8 - 10b_2 = -2$
$x_2 = 8 - x_1 = 10$

To summarize, the relaxed solution is:
$b_1 = 0$, $b_2 = 1$, $x_1 = -2$, $x_2 = 10$, $z = 34$

This is the same solution that we found at the root. Again, the relaxed solution is feasible (substitute into the inequality constraints to confirm). The integer variables take on integer values for the relaxed solution, hence this is the optimal solution for the integer program with b_1=0. We don't need to explore any descendants of this node.
In addition, this is better than the incumbent, hence we update the incumbent to:
$z^* = 34$
$b_1 = 0$, $b_2 = 1$, $x_1 = -2$, $x_2 = 10$

Now we explore the other option for b_1.

Node $b_1 = 1$:

Form the relaxed problem by substituting for b_1 and relaxing the integer constraint::
Minimize $z = 8x_1 + 3x_2 + 18 + 20b_2$
subject to $x_1 + x_2 \geq 8$
$x_1 \leq 15$
$x_2 \leq 10b_2$
$b_2 \leq 0$
$b_2 \in [1,0]$.

First, note that $b_2 \leq 0$ and $b_2 > 0$, hence:

$b_2 = 0$

Substituting for b_2 and simplifying we get:

Minimize $z = 8x_1 + 3x_2 + 18$

subject to $x_1 + x_2 \geq 8$

$x_1 \leq 15$

$x_2 \leq 0$

.

Second, we again note that z monotonically decreases with x1 and x2, thus:

$x_1 + x_2 \geq 8 \quad \rightarrow x_1 + x_2 = 8$ (i.e., the constraint is said to be active)

$x_2 = 8 - x_1$

Substituting for x_2 produces:

Minimize $z = 5x_1 + 42$

subject to $x_1 \leq 15$

$8 \leq x_1$

Finally, z monotonically decreases with x_1, hence z is a minimum when:

$8 \leq x_1 \rightarrow x_1 = 8$,

Substituting for x_1 into the equations for z and x_2 produces:

$z = 5(8) + 42 = 82$

$x_2 = 8 - 8 = 0$

To summarize, the relaxed solution is:

$b_1 = 1, b_2 = 0, x_1 = 8, x_2 = 0, z = 82$

This solution is feasible, and the integer variables take on integer values, hence the node is fathomed. The solution is worse than the incumbent, hence the solution is thrown away. The final solution is:

$z^* = 34$

$b_1 = 0, b_2 = 1, x_1 = -2, x_2 = 10$