Markov Decision Processes:
Reactive Planning to Maximize Reward

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16.410
November 8th, 2004

Slides adapted from:
Manuela Veloso,
Reid Simmons, &
Tom Mitchell, CMU

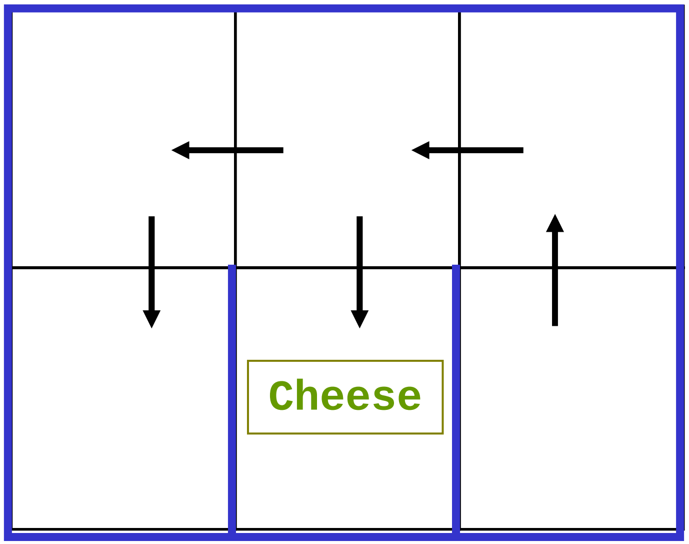
Reading and Assignments

- Markov Decision Processes
  - Read AIMA Chapter 17, Sections 1 – 3.

❖ This lecture based on development in:

"Machine Learning" by Tom Mitchell
Chapter 13: Reinforcement Learning

How Might a Mouse Search a Maze for Cheese?



- State Space Search?
- As a Constraint Satisfaction Problem?
- Goal-directed Planning?
- Linear Programming?

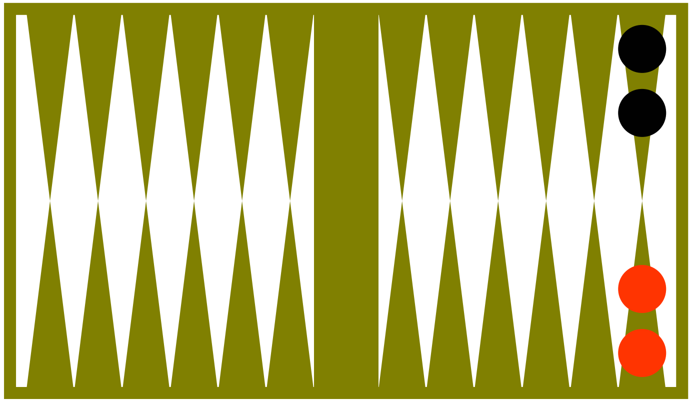
What is missing?

Ideas in this lecture

- Problem is to accumulate rewards,
  rather than to achieve goal states.
- Approach is to generate reactive policies for how to act in all situations, rather than plans for a single starting situation.
- Policies fall out of value functions, which describe the greatest lifetime reward achievable at every state.
- Value functions are iteratively approximated.

MDP Examples: TD-Gammon [Tesauro, 1995]
Learning Through Reinforcement

Learns to play Backgammon



States:

- Board configurations (10^{20})

Actions:

- Moves

Rewards:

  - +100 if win
  - - 100 if lose
  - 0 for all other states

- Trained by playing 1.5 million games against self.

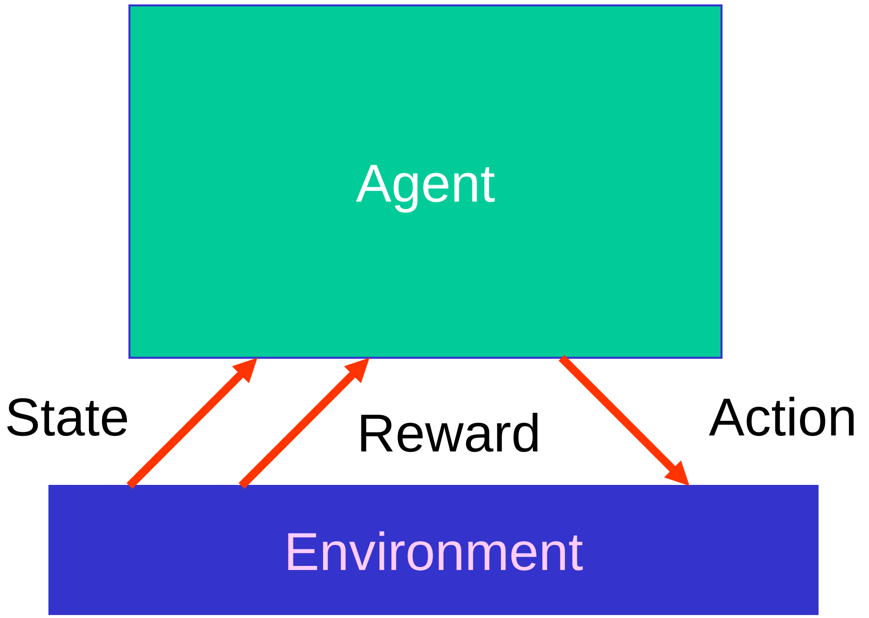
→ Currently, roughly equal to best human player.

MDP Examples: Aerial Robotics [Feron et al.] Computing a Solution from a Continuous Model

Markov Decision Processes

- Motivation
- What are Markov Decision Processes (MDPs)?
  - Models
  - Lifetime Reward
  - Policies
- Computing Policies From a Model
- Summary

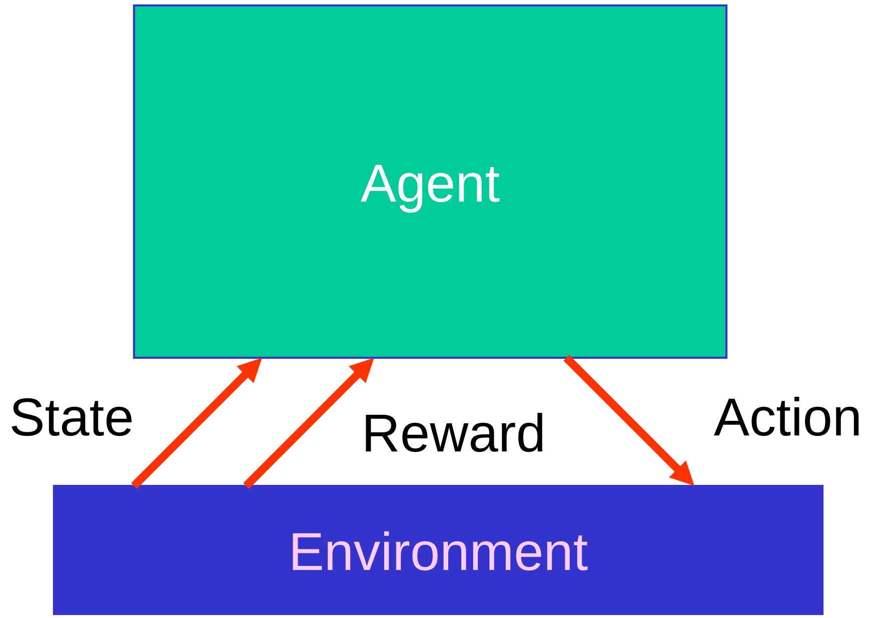
MDP Problem



$$s_0 \xrightarrow[r_0]{a_0} s_1 \xrightarrow[r_1]{a_1} s_2 \xrightarrow[r_2]{a_2} s_3$$

Given an environment model as a MDP create a policy for acting that maximizes lifetime reward

MDP Problem: Model



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Given an environment model as a MDP create a policy for acting that maximizes lifetime reward

Markov Decision Processes (MDPs)

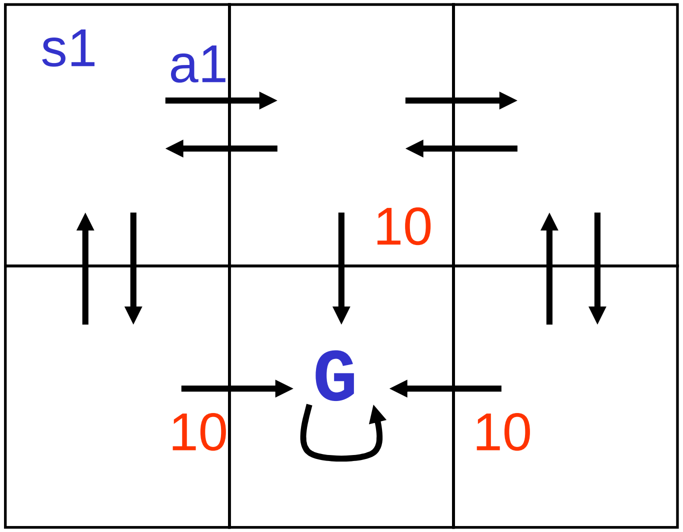
Model:

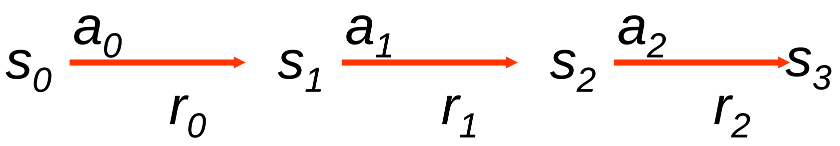
- Finite set of states, *S*
- Finite set of actions, *A*
- (Probabilistic) state transitions, $\delta(s,a)$
- Reward for each state and action, $R(s,a)$

Process:

- Observe state s_t in S
- Choose action a_t in A
- Receive immediate reward r_t
- State changes to s_{t+1}

Example:





- Legal transitions shown
- Reward on unlabeled transitions is 0.

MDP Environment Assumptions

- Markov Assumption:
  Next state and reward is a function only of the current state and action:
  - $s_{t+1} = \delta(s_t, a_t)$
  - $r_t = r(s_t, a_t)$

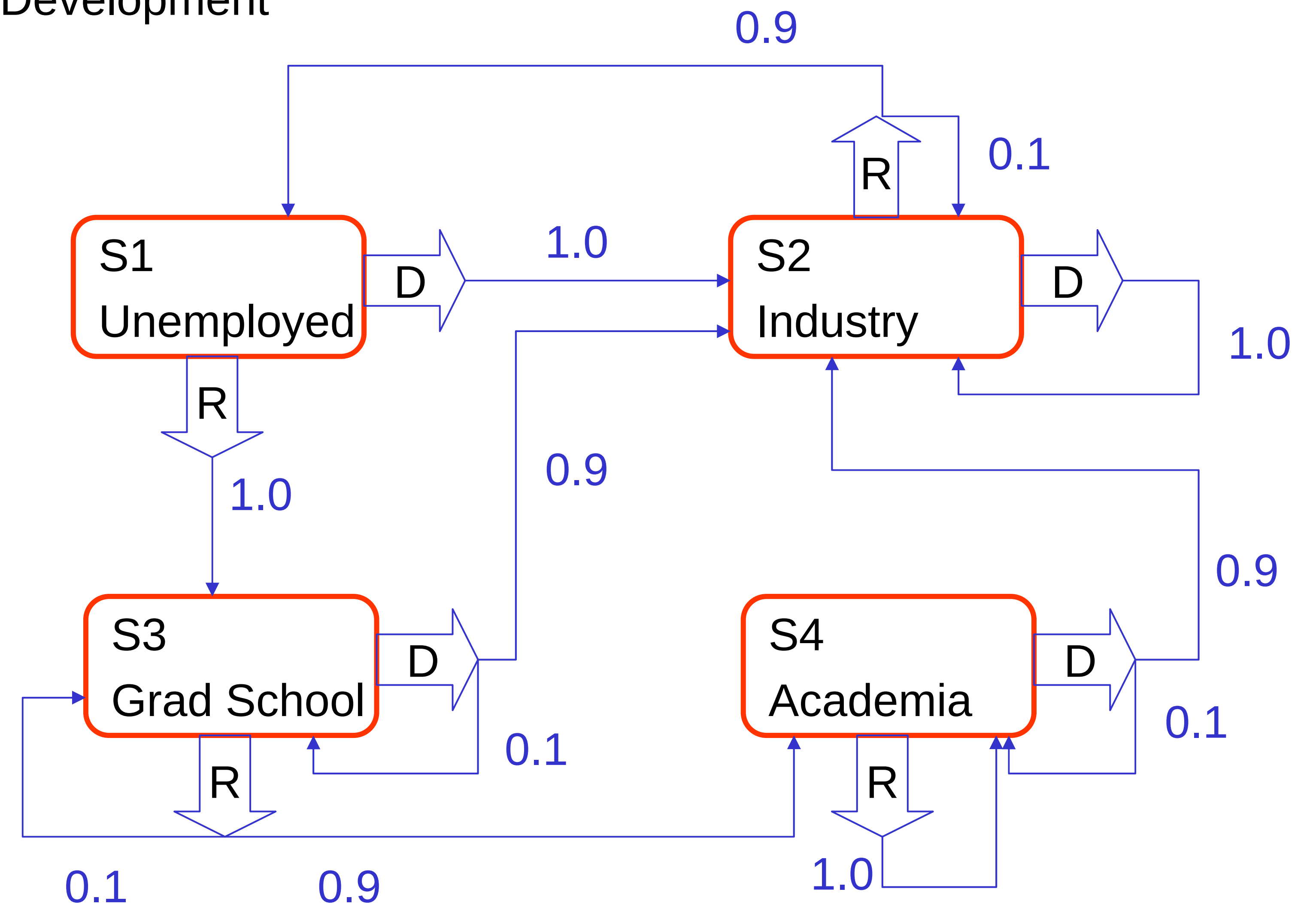
- Uncertain and Unknown Environment:
  δ and r may be nondeterministic and unknown

MDP Nondeterministic Example

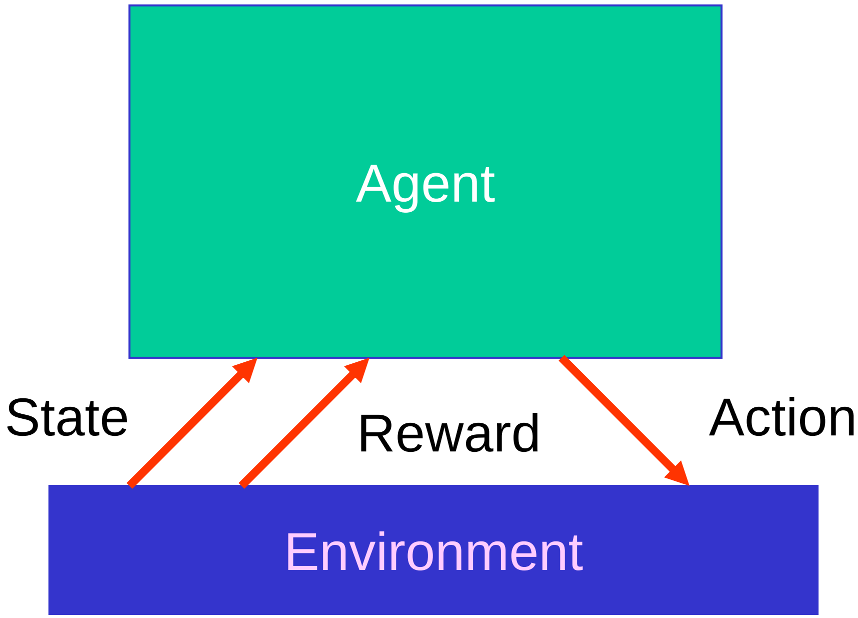
R – Research

D – Development

Today we only consider the deterministic case



MDP Problem: Model



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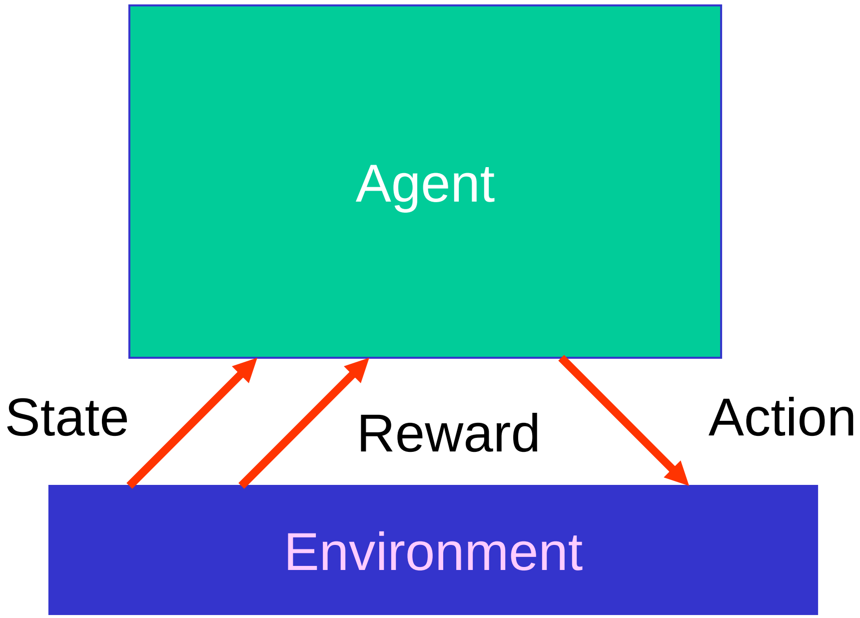
MDP Problem: Lifetime Reward



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Given an environment model as a MDP create a policy for acting that maximizes lifetime reward

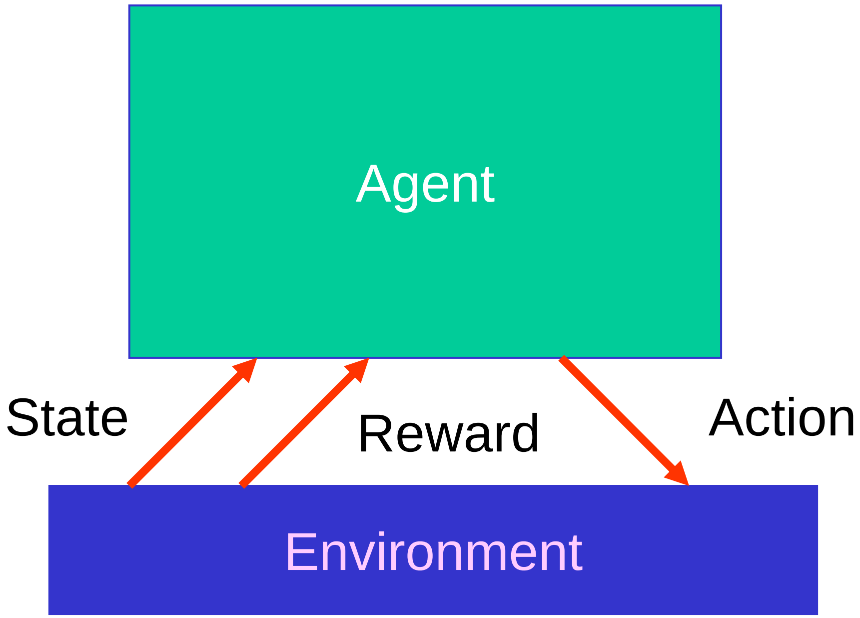
Lifetime Reward

- Finite horizon:
  - Rewards accumulate for a fixed period.
  - \$100K + \$100K + \$100K = \$300K

- Infinite horizon:
  - Assume reward accumulates for ever
  - \$100K + \$100K + . . . = infinity

- Discounting:
  - Future rewards not worth as much
    (a bird in hand …)
  - Introduce discount factor γ
    \$100K + γ \$100K + γ^2 \$100K. . .   converges
  - Will make the math work

MDP Problem: Lifetime Reward

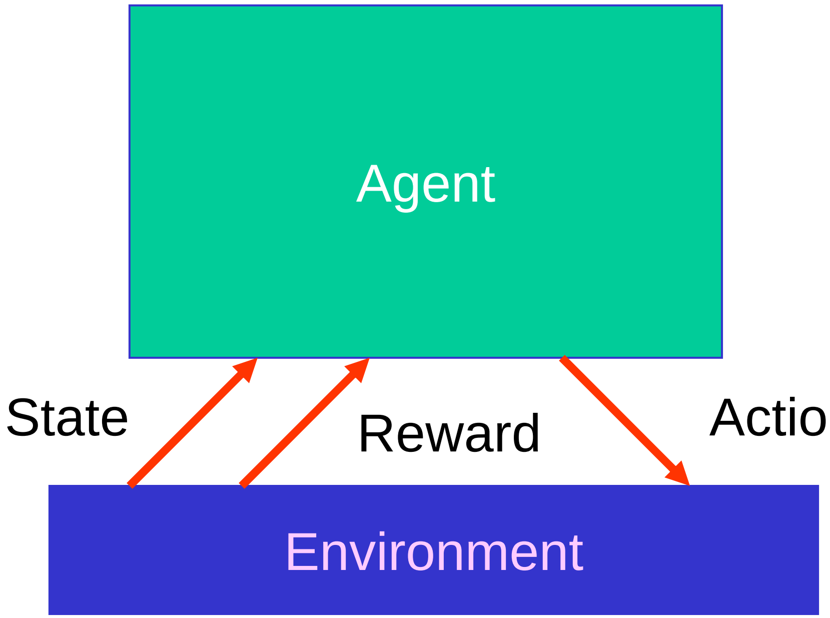


$$s_0 \xrightarrow[r_0]{a_0} s_1 \xrightarrow[r_1]{a_1} s_2 \xrightarrow[r_2]{a_2} s_3$$

Given an environment model as a MDP create a policy for acting that maximizes lifetime reward

$$V = r_0 + \gamma\, r_1 + \gamma^2\, r_2 \;. \;. \;.$$

MDP Problem: Policy



$$s_0 \xrightarrow[r_0]{a_0} s_1 \xrightarrow[r_1]{a_1} s_2 \xrightarrow[r_2]{a_2} s_3$$

Given an environment model as a MDP create a <u>policy</u> for acting that maximizes lifetime reward

$$V = r_0 + \gamma\, r_1 + \gamma^2\, r_2 \ldots$$

Assume deterministic world

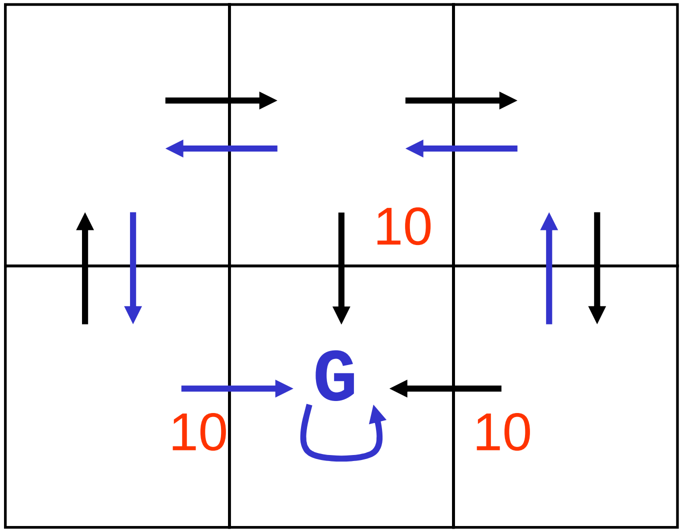
Policy $\pi : S \rightarrow A$

- Selects an action for each state.

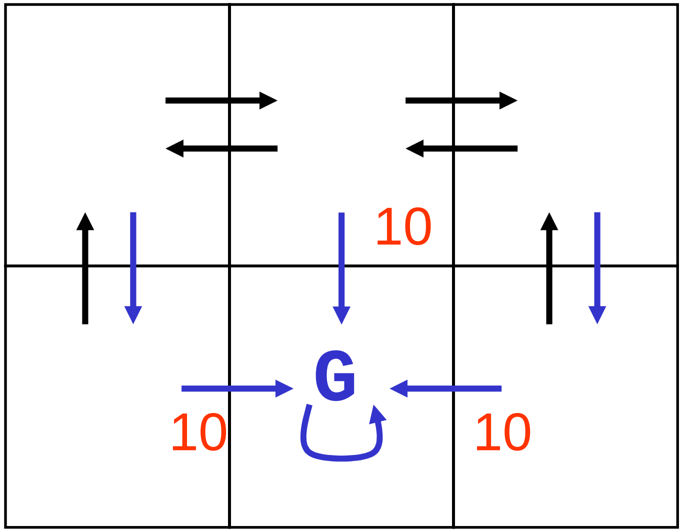
Optimal policy $\pi^* : S \rightarrow A$

- Selects action for each state that maximizes lifetime reward.

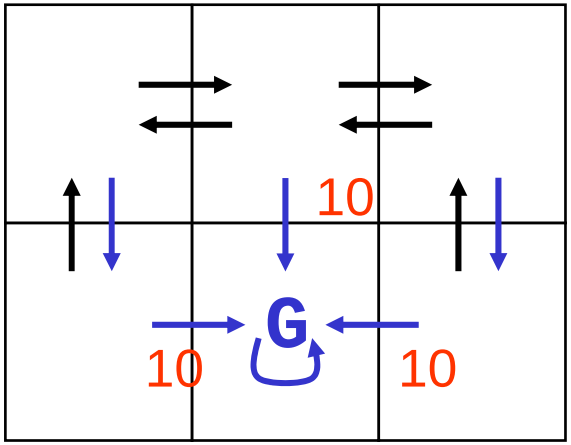
π

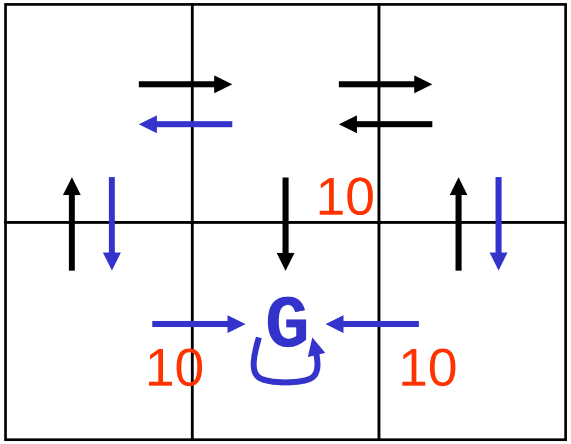


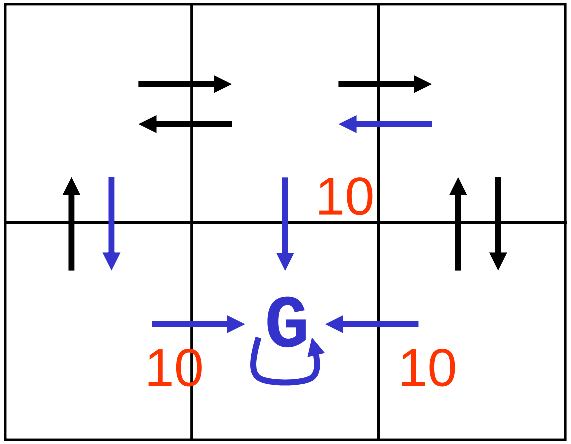
π^*



- There are many policies, not all are necessarily optimal.
- There may be several optimal policies.







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- Computing Policies From a Model
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Markov Decision Processes

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- Computing Policies From a Model
  - Value Functions
  - Mapping Value Functions to Policies
  - Computing Value Functions through Value Iteration
  - An Alternative: Policy Iteration (appendix)
- Summary

Value Function V^{π} for a Given Policy π

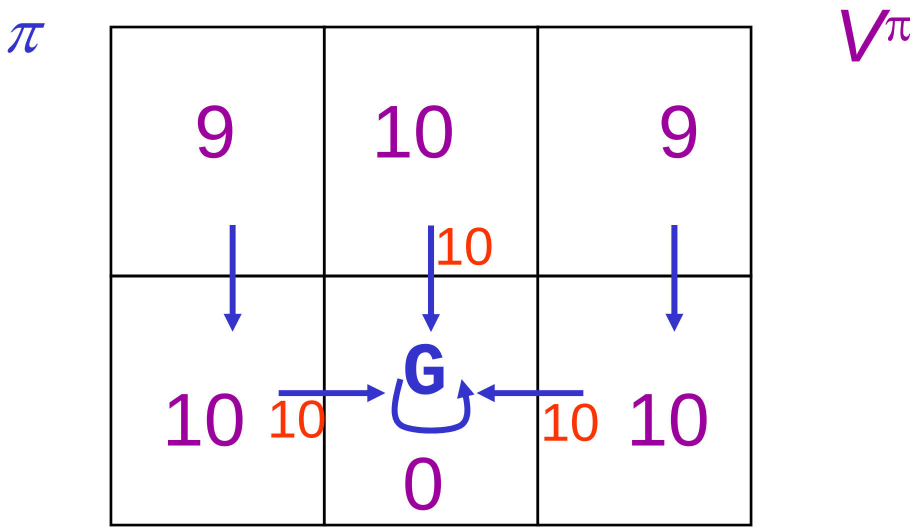
- $V^{\pi}(s_t)$ is the accumulated lifetime reward resulting from starting in state s_t and repeatedly executing policy π:

$$V^{\pi}(s_t) = r_t + \gamma\, r_{t+1} + \gamma^{2}\, r_{t+2} \ldots$$

$$V^{\pi}(s_t) = \sum_i \gamma^{i}\, r_{t+i}$$

where r_t, r_{t+1}, r_{t+2} . . . are generated by following π, starting at s_t .

Assume $\gamma = .9$



An Optimal Policy π* Given Value Function *V**

Idea: Given state s

1. Examine **all** possible actions a_i in state s.
2. Select action a_i with greatest lifetime reward.

Lifetime reward $Q(s, a_i)$ is:

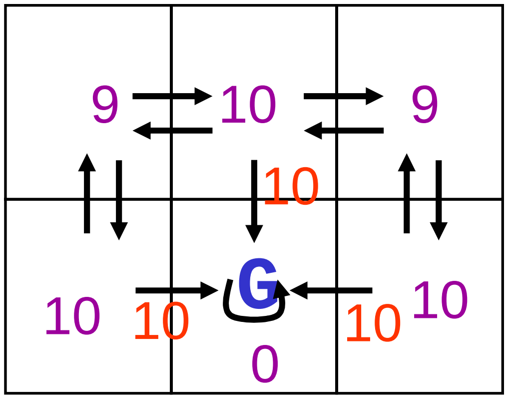
- the immediate reward for taking action *r(s,a)* …
- plus life time reward starting in target state $V(\delta(s, a))$ …
- discounted by γ.

$$\pi^*(s) = \text{argmax}_a\,[r(s,a) + \gamma V^*(\delta(s, a))]$$

Must Know:

- Value function
- Environment model.
    - $\delta : S \times A \to S$
    - $r : S \times A \to \Re$

π

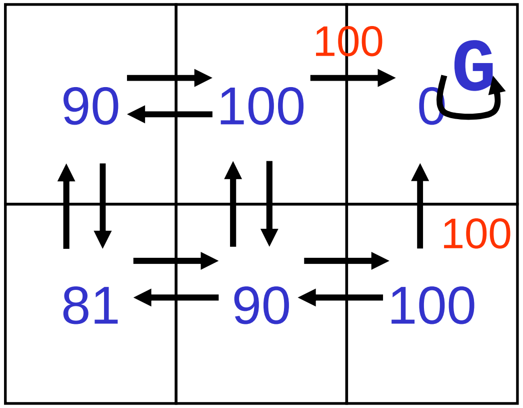


Example: Mapping Value Function to Policy

- Agent selects optimal action from *V*:

$$\pi(s) = \text{argmax}_a\,[r(s,a) + \gamma V(\delta(s, a))]$$

Model + V:



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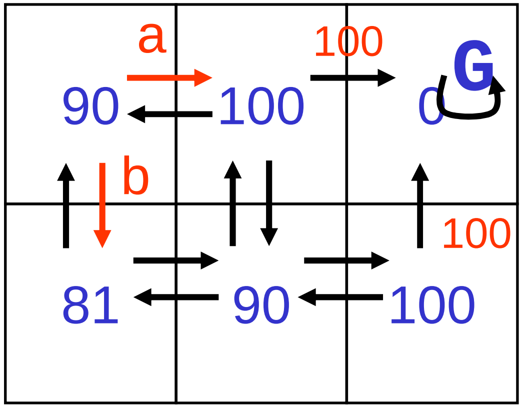
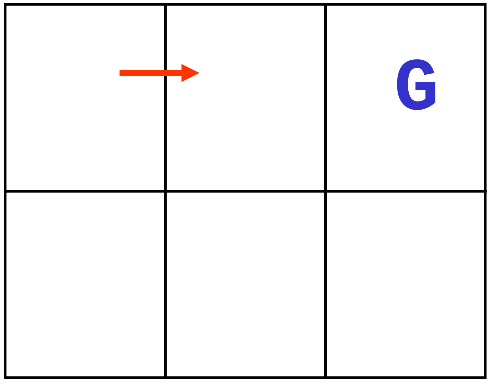


$\gamma = 0.9$

- a: 0 + 0.9 x 100 = 90
- b: 0 + 0.9 x 81 = 72.9
- ➢ select a

π:



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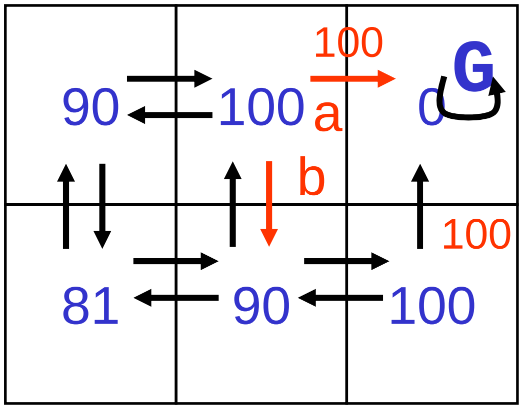

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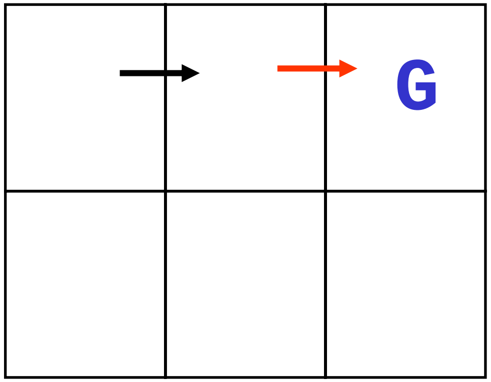
π:



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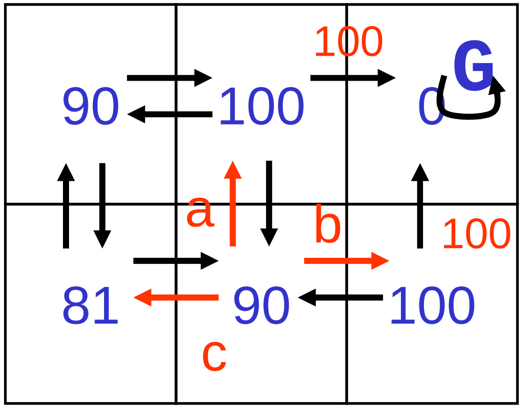
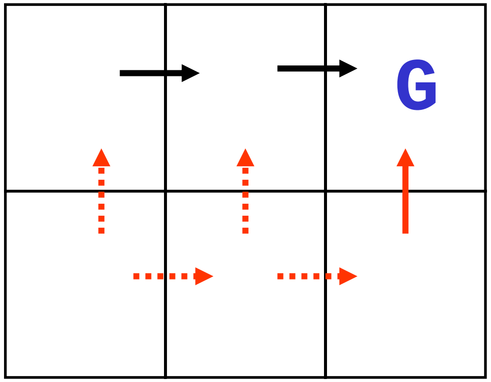


$\gamma = 0.9$

- a: ?
- b: ?
- c: ?
- ➢ select ?

π:



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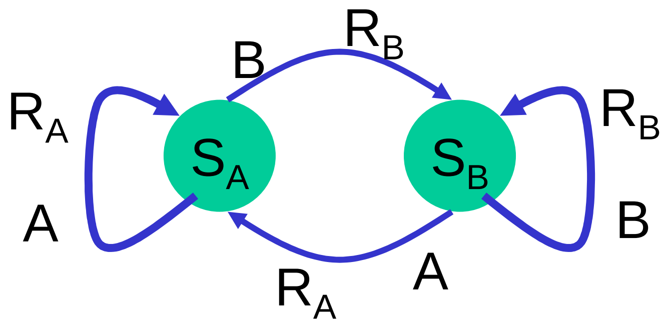
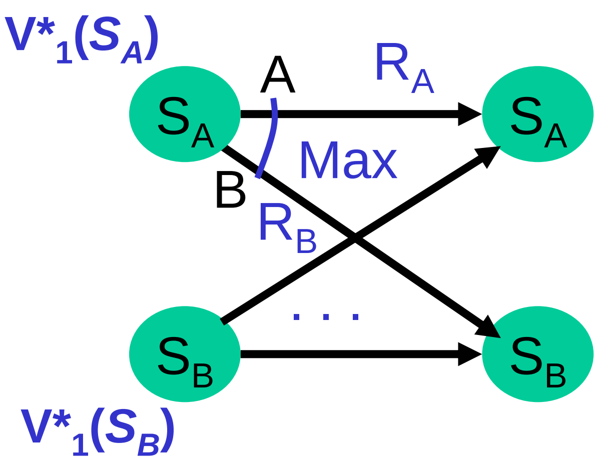
Value Function *V** for an optimal policy π*

Example



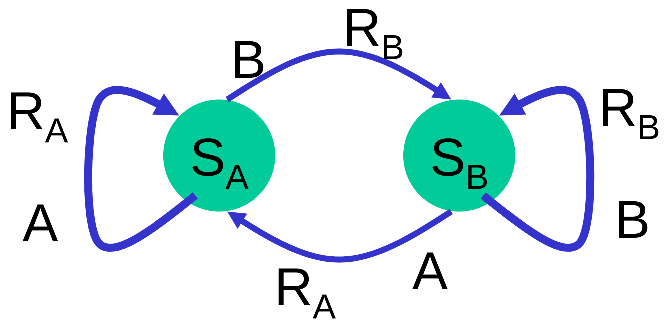
- Optimal value function for a one step horizon:

$$V^*_1(s) = \max_{a_i} [r(s,a_i)]$$



Value Function V^* for an optimal policy π^*

Example



- Optimal value function for a one step horizon:

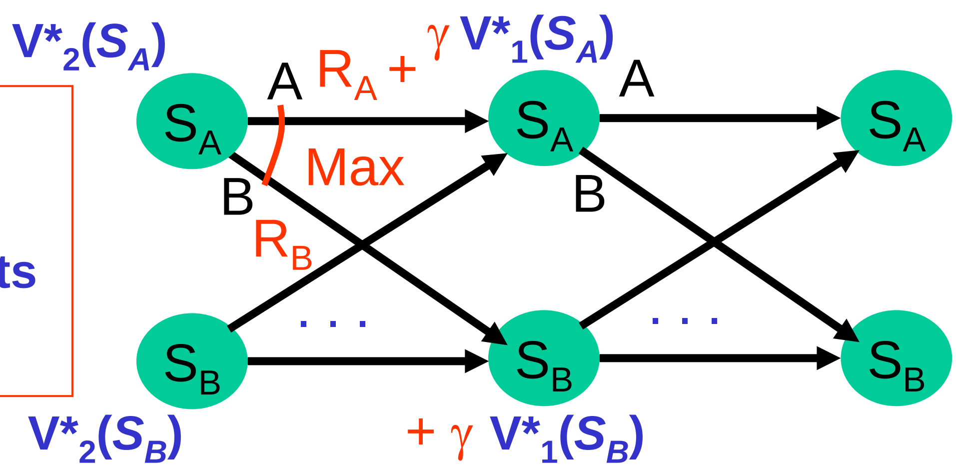
$$V^*_1(s) = \max_{a_i} [r(s,a_i)]$$

- Optimal value function for a two step horizon:

$$V^*_2(s) = \max_{a_i} [r(s,a_i) + \gamma V_1^*(\delta(s, a_i))]$$

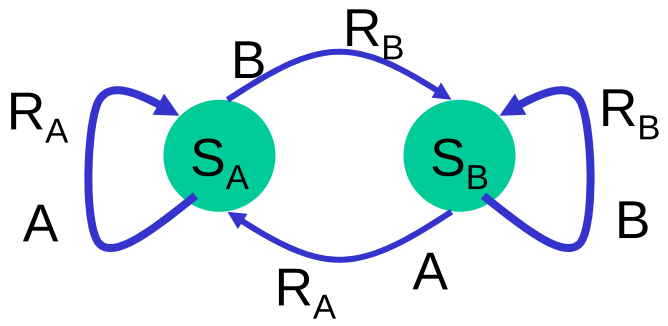
Instance of the Dynamic Programming Principle:

- **Reuse shared sub-results**
- **Exponential saving**



Value Function *V** for an optimal policy π^*

Example



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- Optimal value function for an n step horizon:

$$V^*_n(s) = \max_{a_i} [r(s,a_i) + \gamma V_{n-1}^*(\delta(s, a_i))]$$

Value Function V^* for an optimal policy π^*

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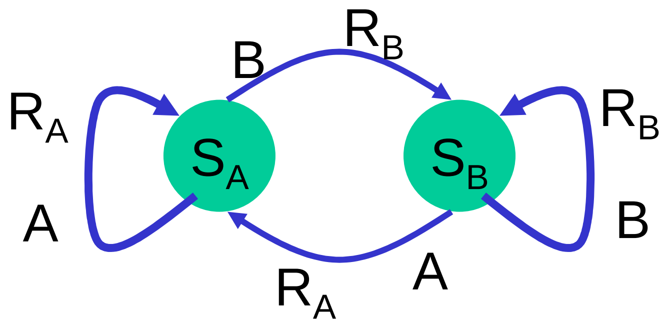

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- Optimal value function for an n step horizon:

$$V^*_n(s) = \max_{a_i} [r(s,a_i) + \gamma V_{n-1}^*(\delta(s, a_i))]$$

➢ Optimal value function for an infinite horizon:

$$\boxed{V^*(s) = \max_{a_i} [r(s,a_i) + \gamma V^*(\delta(s, a_i))]}$$

Solving MDPs by Value Iteration

Insight: Can calculate optimal values iteratively using Dynamic Programming.

Algorithm:

- Iteratively calculate value using Bellman's Equation:

$$V^*_{t+1}(s) \leftarrow \max_a [r(s,a) + \gamma V^*_t(\delta(s, a))]$$

- Terminate when values are "close enough"

$$|V^*_{t+1}(s) - V^*_t(s)| < \varepsilon$$

- Agent selects optimal action by one step lookahead on V^*:

$$\pi^*(s) = \text{argmax}_a [r(s,a) + \gamma V^*(\delta(s, a))]$$

Convergence of Value Iteration

- If terminate when values are "close enough"

$$|V_{t+1}(s) - V_t(s)| < \varepsilon$$

Then:

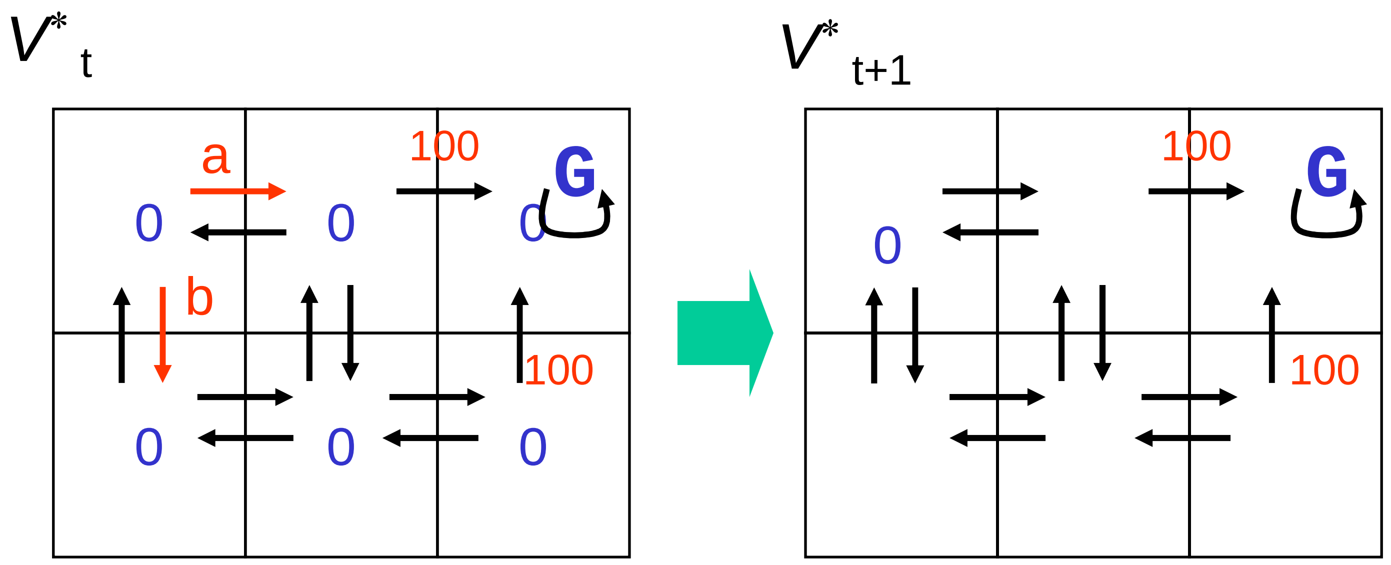
$$\text{Max}_{s \text{ in } S} |V_{t+1}(s) - V^*(s)| < 2\varepsilon\gamma/(1 - \gamma)$$

- Converges in polynomial time.
- Convergence guaranteed even if updates are performed infinitely often, but asynchronously and in any order.

Example of Value Iteration

$$V^*_{t+1}(s) \leftarrow \max_a [r(s,a) + \gamma V^*_t(\delta(s, a))]$$

$\gamma = 0.9$



- a: 0 + 0.9 x 0 = 0
- b: 0 + 0.9 x 0 = 0
- ➢ Max = 0

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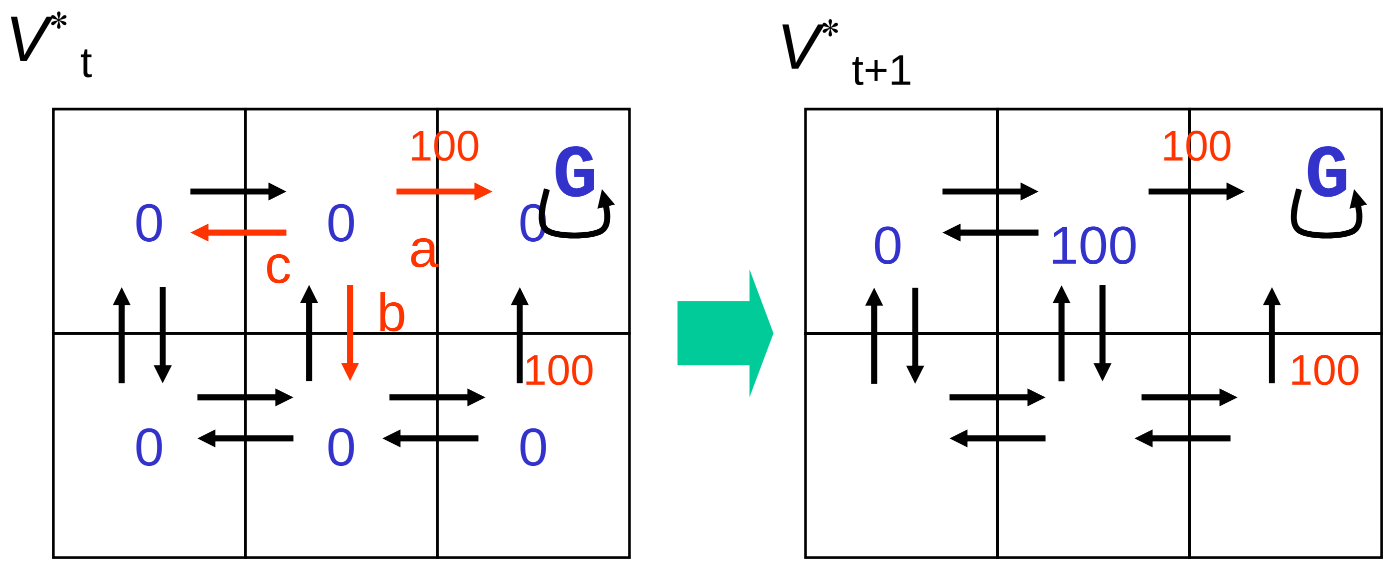

- a: 100 + 0.9 x 0 = 100
- b: 0 + 0.9 x 0 = 0
- c: 0 + 0.9 x 0 = 0

➢ Max = 100

Example of Value Iteration

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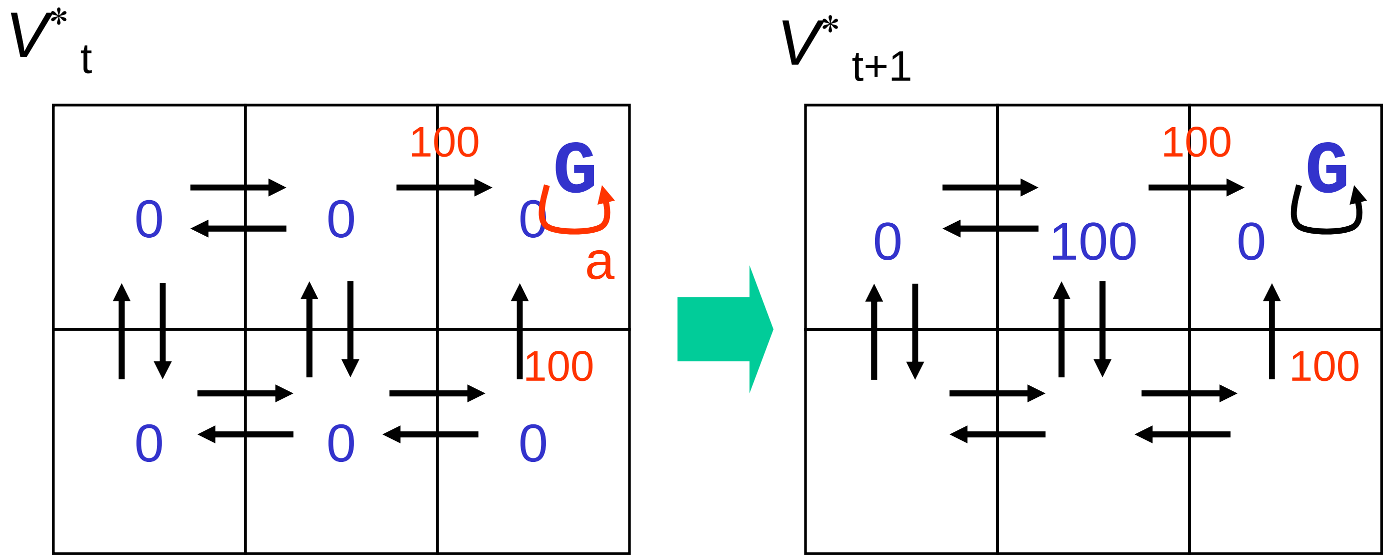

- a: 0 + 0.9 x 0 = 0
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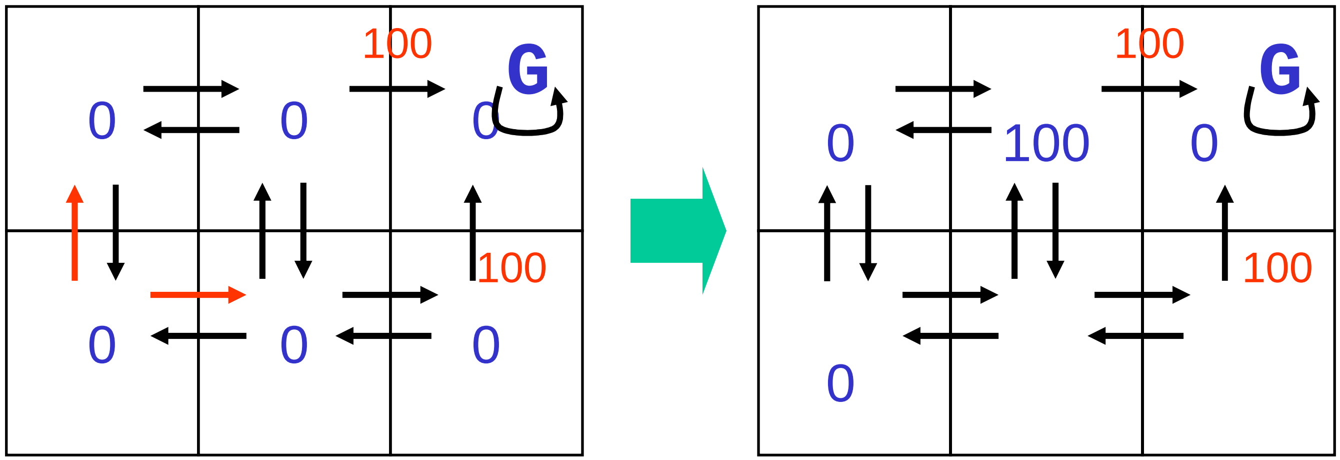
V^*_t V^*_{t+1}



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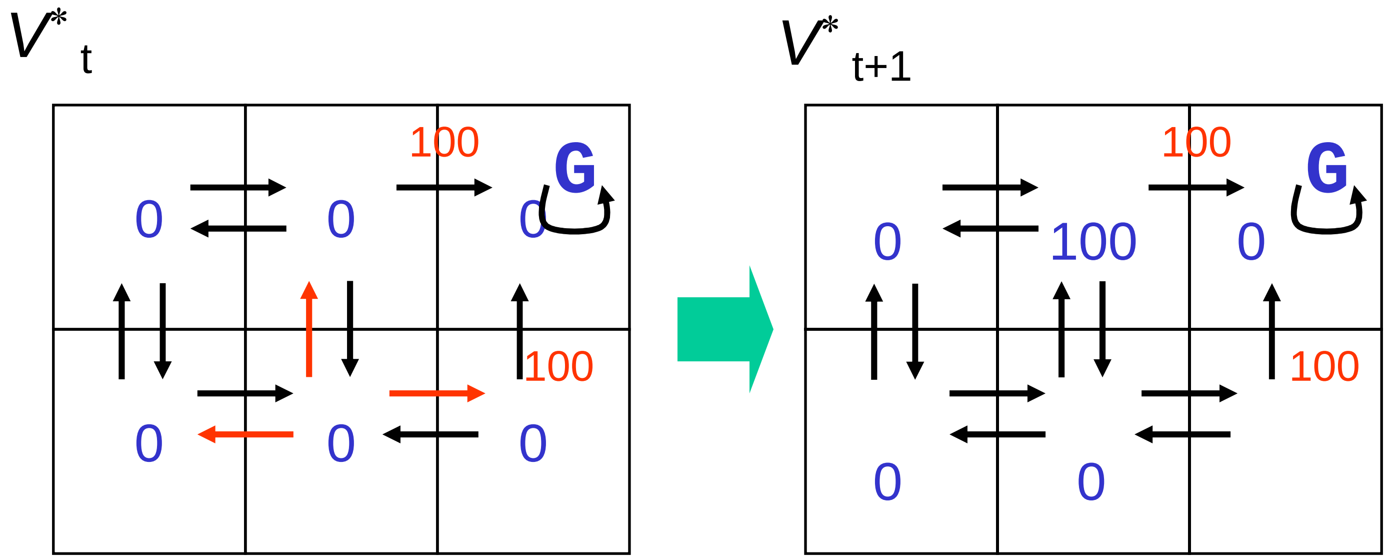

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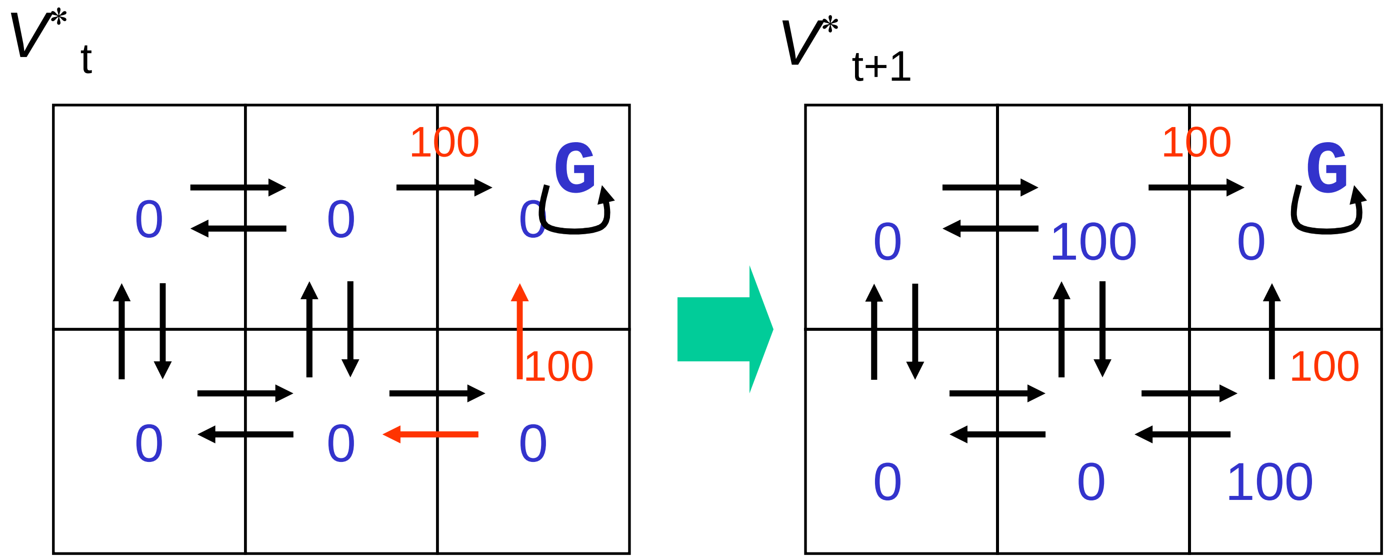

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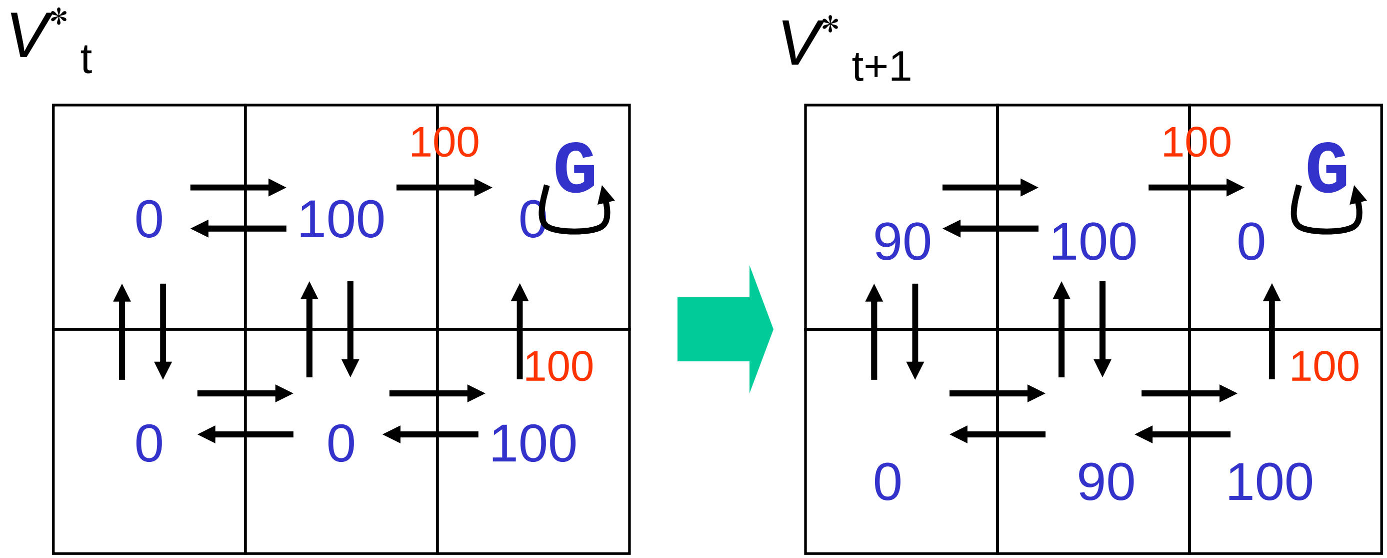

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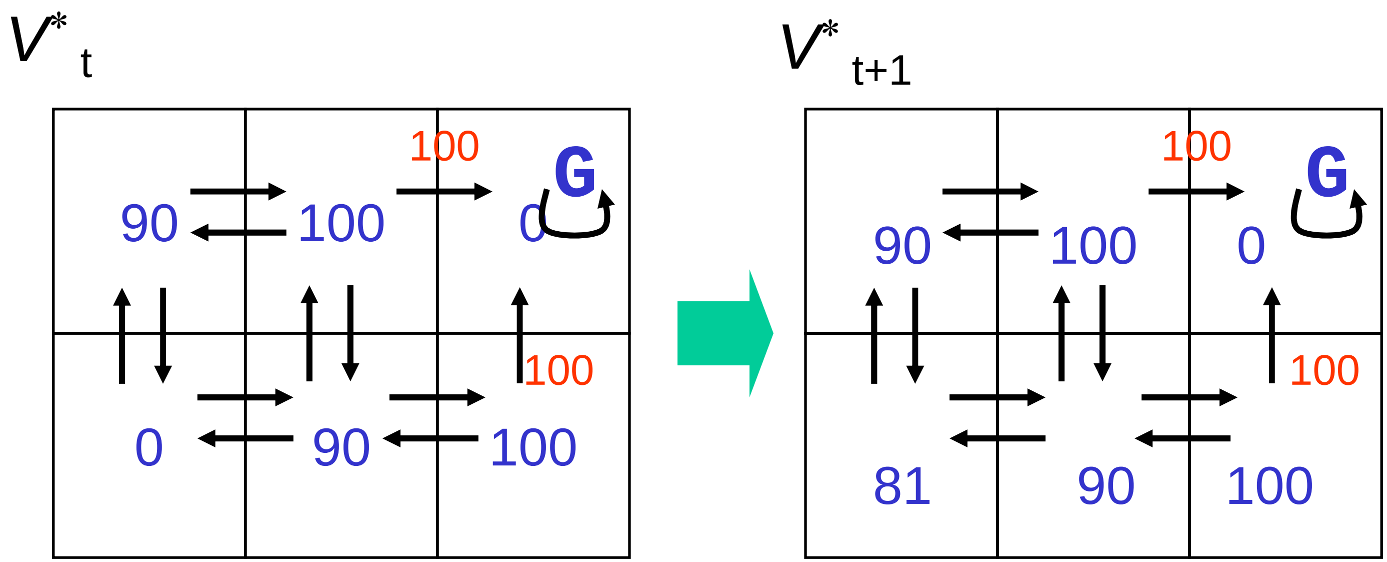

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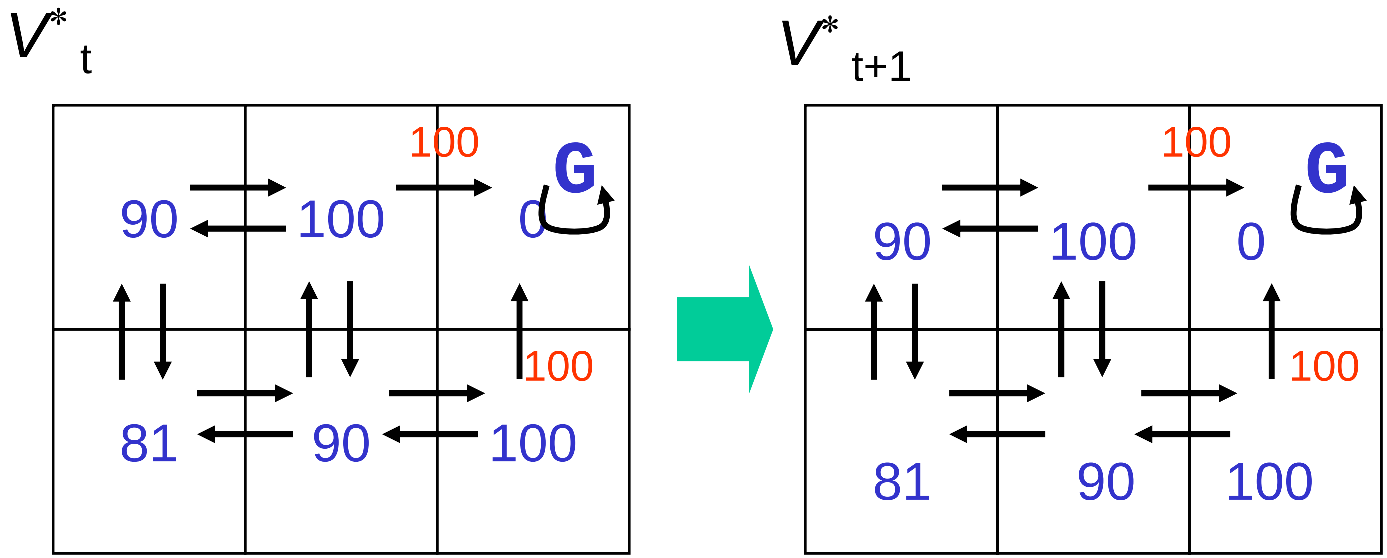

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Appendix: Policy Iteration

Idea: Iteratively improve the policy

1. Policy Evaluation: Given a policy π_i calculate $V_i = V^{\pi i}$, the utility of each state if π_i were to be executed.
2. Policy Improvement: Calculate a new maximum expected utility policy π_{i+1} using one-step look ahead based on V_i.

- π_i improves at every step, converging if $\pi_i = \pi_{i+1}$.
- Computing V_i is simpler than for Value iteration (no max):

$$V^*_{t+1}(s) \leftarrow r(s, \pi_i(s)) + \gamma V^*_t(\delta(s, \pi_i(s)))]$$

  - Solve linear equations in $O(N^3)$
  - Solve iteratively, similar to value iteration.

Markov Decision Processes

- Motivation
- Markov Decision Processes
- Computing policies from a model
  - Value Iteration
  - Policy Iteration
- Summary

Markov Decision Processes (MDPs)

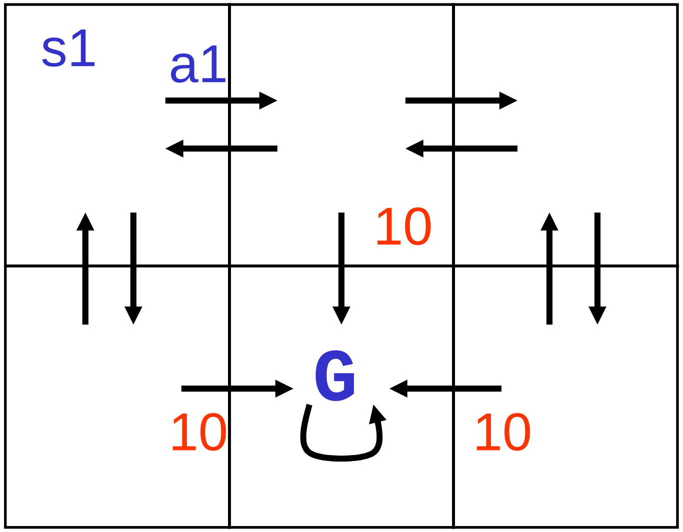
Model:

- Finite set of states, *S*
- Finite set of actions, *A*
- Probabilistic state transitions, $\delta(s,a)$
- Reward for each state and action, $R(s,a)$

Process:

- Observe state s_t in S
- Choose action a_t in A
- Receive immediate reward r_t
- State changes to s_{t+1}

Deterministic Example:



$$s_0 \xrightarrow[r_0]{a_0} s_1 \xrightarrow[r_1]{a_1} s_2 \xrightarrow[r_2]{a_2} s_3$$

Crib Sheet: MDPs by Value Iteration

Insight: Can calculate optimal values iteratively using Dynamic Programming.

Algorithm:

- Iteratively calculate value using Bellman's Equation:

$$V^*_{t+1}(s) \leftarrow \max_a [r(s,a) + \gamma V^*_t(\delta(s, a))]$$

- Terminate when values are "close enough"

$$|V^*_{t+1}(s) - V^*_t(s)| < \varepsilon$$

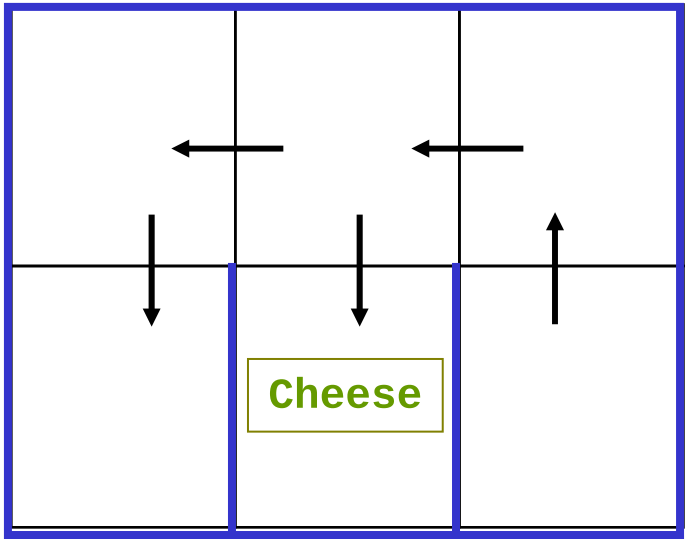
- Agent selects optimal action by one step lookahead on V^*:

$$\pi^*(s) = \operatorname{argmax}_a [r(s,a) + \gamma V^*(\delta(s, a)]$$

Ideas in this lecture

- Objective is to accumulate rewards, rather than goal states.
- Objectives are achieved along the way, rather than at the end.
- Task is to generate policies for how to act in all situations, rather than a plan for a single starting situation.
- Policies fall out of value functions, which describe the greatest lifetime reward achievable at every state.
- Value functions are iteratively approximated.

How Might a Mouse Search a Maze for Cheese?



- By Value Iteration?
- What is missing?