

Introduction to Linear Programming

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Historical aspects

- Historical contributor: G. Dantzig, late 1940's (Now at Stanford University. Realize many real-world design problems can be formulated as linear programs and solved efficiently. Finds algorithm, the Simplex method to solve LP's. As of 1997, still best algorithm for most applications.
- So important for world economy that any new algorithmic development on LP's is likely to make the Front Page of major newspapers (e.g. NY times, Wall Street Journal). Example: 1979 L. Khachyan's adaptation of ellipsoid algorithm, N. Karmarkar's new interior-point algorithm.
- A remarkably practical and theoretical framework: LPs eat a large chunk of total scientific computational power expended today. It is crucial for economic success of most distribution/transport industries and to manufacturing.
- Now becomes suitable for real-time applications.

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Examples of Linear programs

•Example 1: Revenue maximization problem.

Wage is \$8 / hour

Stock Profit is $\text{daily \% yield} = \frac{\text{\# hours spent}}{30} \times \text{capital}$

and number of hours spent is never greater than 10 hours/day
(3 hours Max on stock market)

- Call x_1 the number of hours spent @ work, x_2 the number of hours spent working on the stock market.

•Problem formulation:

$$\text{Maximize } 8x_1 + \frac{x_2 \times \text{capital}}{3000}$$

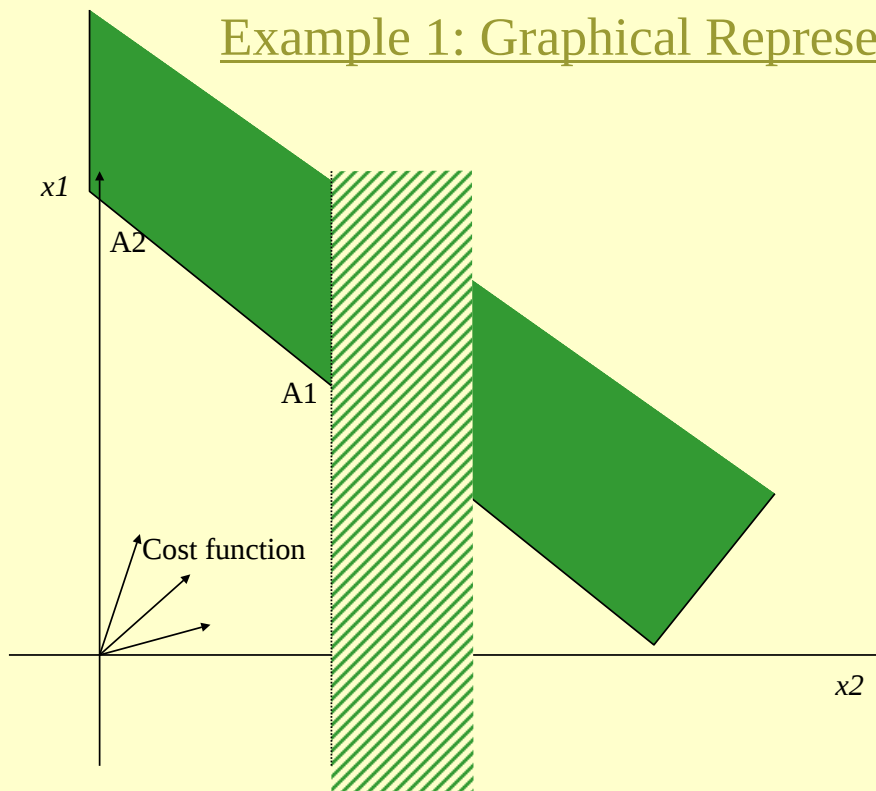
$$\text{Subject to } x_1 + x_2 \leq 10$$

$$x_2 \leq 3$$

Our first LP

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Example 1: Graphical Representation



Answer depends upon gradient orientation

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Example 2: Resource Allocation

An aircraft company makes two aircraft engines. Engine parts are made at three plants: A, B, C.

- Engine 1 is made of elements produced in A and C
- Engine 2 is made of elements of B and C

- Plants A,B, C have limited throughput
 - What mix of product 1 and 2 will bring maximum profit?
- Another resource allocation problem.

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Example 2 Ct'd

- Add'l data:

Plant	Production time per engine 1	Production time per engine 2	Production time Available Per week
A	1	0	4
B	0	2	12
C	3	2	18
Profit/engine	\$300,000	\$500,000	

x_1 : # of Engine 1

x_2 : # Engine 2

Profit (in KKK\$): $3x_1 + 5x_2$

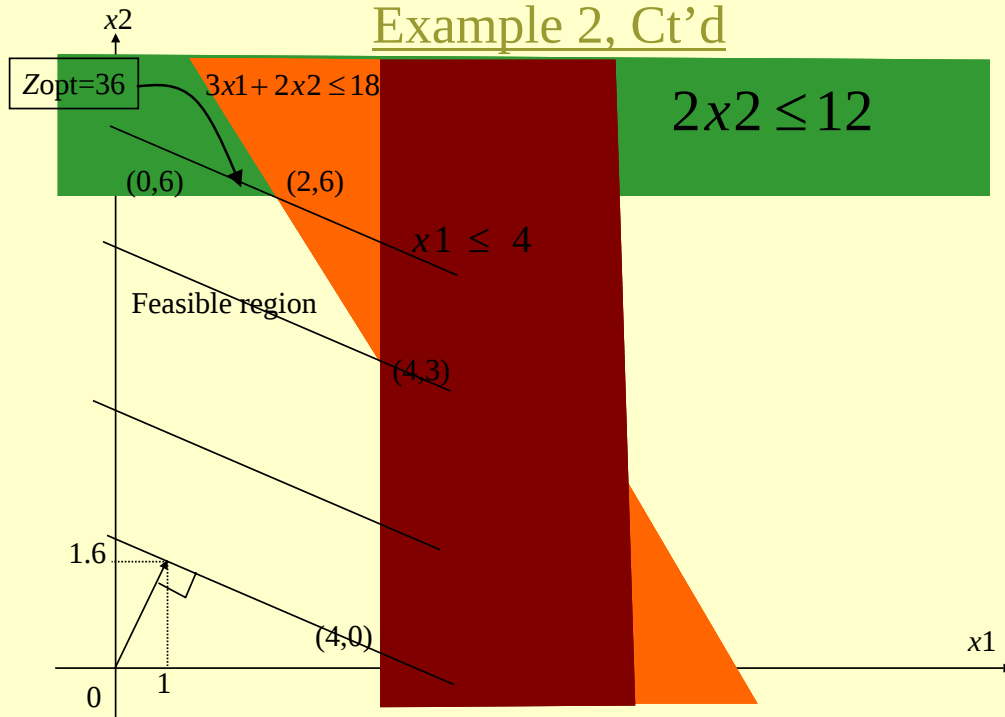
Constraints: $x_1 \leq 4$

$$2x_2 \leq 12$$

$$3x_1 + 2x_2 \leq 18$$

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Example 2, Ct'd



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Standard Linear Programming Model

- Notations:
 - Z : Overall measure of performance
 - x_j : Activity level for product j
 - c_j : Marginal improvement of Z associated with product j
 - b_i : Available resource I
 - a_{ij} : Amount of resource I to produce j .
- General LP: Maximize $Z = c_1x_1 + \dots + c_nx_n$
 Subject to $a_{11}x_1 + \dots + a_{1n}x_n \leq b_1$
 $\vdots$
 $a_{m1}x_1 + \dots + a_{mn}x_n \leq b_m$
 $x_1 \geq 0, \dots, x_n \geq 0$.
- Standard form for LP. Minimize $Z = -c_1x_1 - \dots - c_nx_n$

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Standard form of LP

• Other forms of LPs:

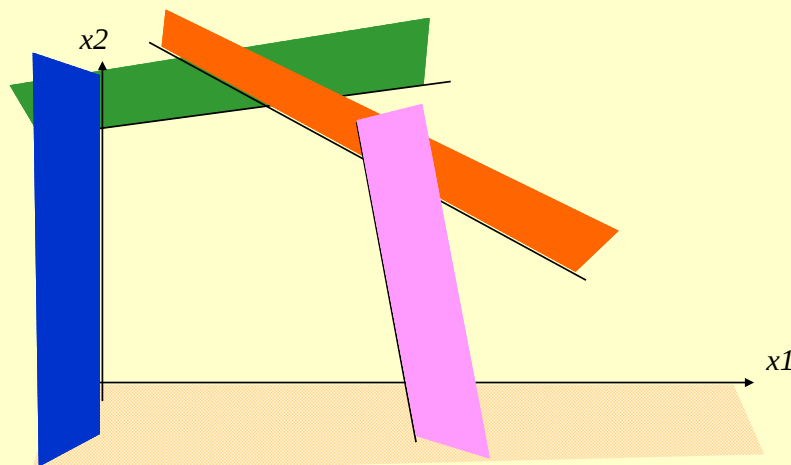
- Arbitrary signs on c_i , b_j , a_{ij} , x_i .
- Equality constraints
- Require technicalities but may be cast in standard form

• Feasible/unfeasible solutions

The set of solutions that satisfy all the constraints is said feasible. It is the intersection of many half planes (m of them) with the *positive orthant* (what's that?). It's a polytope.

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Graphical Depiction of LP constraints



$x_1 \geq 0$ and $x_2 \geq 0$

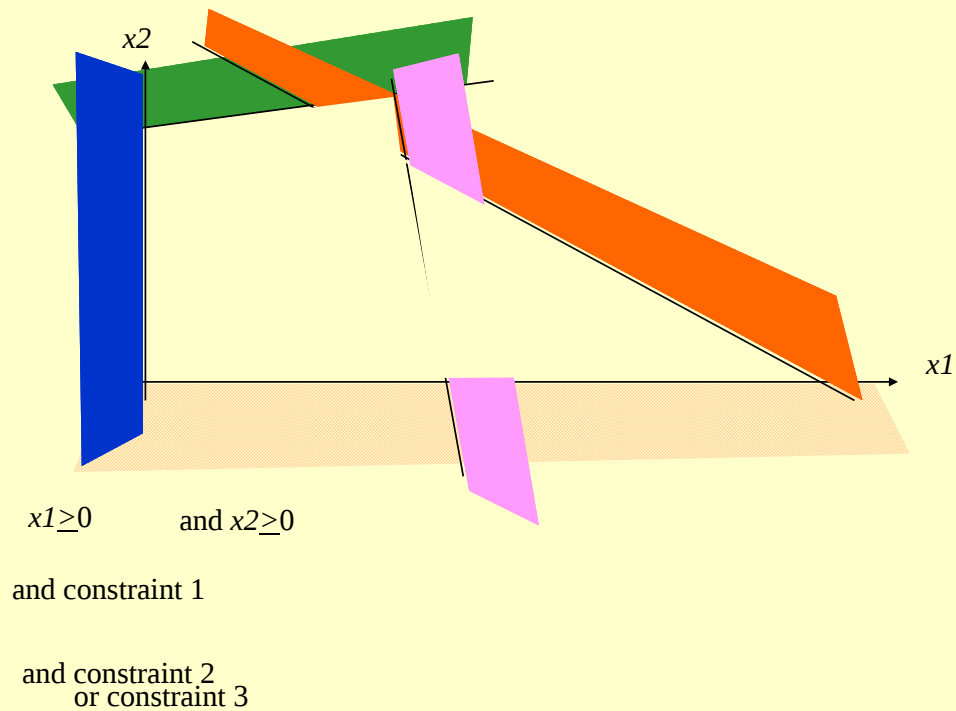
and constraint 1

and constraint 2

and constraint 3

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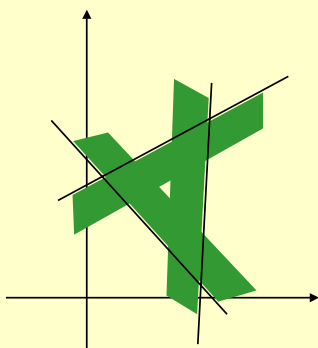
This is not an LP!



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Some vocabulary (ct'd)

- Vocabulary
 - Polytope may be empty: No feasible solutions
 - Polytope may be unbounded: possibility for infinite optimal costs
 - Sure signs of problem misunderstanding



Infeasible set of constraints
(Nonstandard LP)

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LP Assumptions

- Proportionality
 - Does twice the activity result in twice the profit, or twice the shipment size result in twice the cost?
- Additivity
 - Does opening a store in Cambridge and one in Boston result in the sum of projected profits?
- Divisibility
 - Can you manufacture half an aircraft?
- Certainty
 - What do we really know about the problem?

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Addressing these issues

- Divisibility is a real problem in decision-making LPs; integer programming is the solution. Some important integer programs luckily are also solvable as LPs!
 - Theorem: The solution to any transportation problem with integer supplies and demands is integer.
 - Transportation problems include the class of assignment problems.
- Uncertainty in the data is a huge problem. The right way to address it is either by sensitivity analysis if the data might be off by small amounts or by multiple case analysis if the scenarios range widely.

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SIMPLEX method of solving LPs

- Maximize $3x_1 + 5x_2$,
subject to
 $x_1 \leq 4$
 $2x_2 \leq 12$
 $3x_1 + 2x_2 \leq 18$
 $x_1 \geq 0, x_2 \geq 0$
- Standard (=) form:

$$\begin{array}{rcl} \min z - 3x_1 - 5x_2 & & = 0 \\ x_1 & + x_3 & = 4 \\ x_2 & + x_4 & = 6 \\ 3x_1 + 2x_2 & + x_5 & = 18 \\ x_1, x_2, x_3, x_4, x_5 & \geq & 0 \end{array}$$

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Basis solution to a system of equations

- Standard (=) form:

$$\begin{array}{rcl} \min z - 3x_1 - 5x_2 & & = 0 \\ x_1 & + x_3 & = 4 \\ x_2 & + x_4 & = 6 \\ 3x_1 + 2x_2 & + x_5 & = 18 \end{array}$$
- The variables whose coefficients are an identity matrix are x_3, x_4, x_5 . These are slack variables, and at this step in the Simplex method they are also the basic variables
- The other variables are nonbasic and are set to zero at this corner point. Let $x_1=0, x_2=0$.
- So $x_3=4, x_4=6$, and $x_5=18$. At each basic solution you can read the values of basic variables off the equations

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Pivot step

- Choose most negative reduced cost: $z -3x_1 -5x_2$
- So pivot on column x_2
- Choose smallest positive ratio test:

$$\begin{array}{rclcl}
 x_1 & +x_3 & = 4 & \text{blank} \\
 & x_2 & +x_4 & = 6 & \mathbf{6 \div 1 = 6} \\
 3x_1+2x_2 & & +x_5 = 18 & & 18 \div 2 = 9
 \end{array}$$

- So pivot on row 2, which means: set the coefficient of x_2 in row 2 to 1, and eliminate x_2 from the objective and the other constraint rows

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Pivot step

- Choose most negative reduced cost: $z -3x_1 +5x_4 = -30$
- So pivot on column x_1
- Choose smallest positive ratio test:

$$\begin{array}{rclcl}
 x_1 & +x_3 & = 4 & 4 \div 1 = 4 \\
 & x_2 & +x_4 & = 6 & \text{blank} \\
 3x_1 & & -2x_4+x_5 = 6 & & \mathbf{6 \div 3 = 2}
 \end{array}$$

- So pivot on row 3, which means: set the coefficient of x_1 in row 3 to 1, and eliminate x_1 from the objective and the other constraint rows

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Pivot step

- Choose most negative reduced cost: $z = -36 + 3x_4 + x_5 = -36$
- No more improvement in z possible because all reduced costs are positive
- Read off solution:

$$\begin{array}{rcl} x_3 + 2/3x_4 + 1/3x_5 & = & 4 \\ x_2 + x_4 & = & 6 \\ x_1 - 2/3x_4 + 1/3x_5 & = & 2 \end{array}$$

- $z = -36$, so original objective is 36. $x_1=2, x_2=6, x_3=4$
- Positive slack variable x_3 indicates that original constraint $x_1 \leq 4$ is not tight; the other two constraints are tight

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Simplex algorithm steps

- Convert to standard form
- Add slack variables
- Pivot
 - choose most negative reduced cost (identifies new basic variable)
 - choose lowest positive ratio in ratio test
 - eliminate new basic variable from objective and all rows except lowest positive ratio in ratio test row
- Stop if optimal and read solution. Otherwise, pivot again.

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Simplex algorithm analysis

- Does every pivot step decrease cost? Answer: almost yes!

$$\begin{array}{rclcl}
 \text{Min:z} & -8x_4 & & = 20 & \\
 & x_3 + 2/3x_4 + 1/3x_5 & = 4 & 4 \div 2/3 = 6 & \\
 & x_2 & + x_4 & = 0 & 0 \div 1 = 0 \\
 & x_1 & - 2/3x_4 + 1/3x_5 & = 2 & 2 \div -2/3 \text{ Not valid}
 \end{array}$$

- Cost has not decreased! This is a **degenerate basis**, with one basic variable (x_4) having value zero

$$\begin{array}{rclcl}
 \text{Min:z} & +8x_2 & & = 20 & \\
 & -2/3x_2 + x_3 & + 1/3x_5 & = 4 & \\
 & x_2 & + x_4 & = 0 & \\
 & x_1 + 2/3x_2 & & + 1/3x_5 & = 2
 \end{array}$$

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Simplex algorithm analysis

- If there are no degenerate bases (true in random instances of LP), then it is impossible to return to a basis that has already been visited, because the objective function must decrease at every step
- If there are degenerate bases, it is possible to **cycle**, which would mean that the same series of bases repeats as you pivot, so one could pivot forever.
- Smart rules about disambiguating ties in the lowest cost coefficient test and minimum positive ratio test can rule out possibility of infinite cycling
- Number of pivot steps $\leq$ Number of corner point solutions

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Simplex algorithm analysis

- With n variables, each bounded ≥ 0 , ≤ 1 , there are 2^n corner point feasible solutions
- Pathological examples force examination of all of these
- Each pivot (Gaussian elimination) takes $m n$ time
- Simplex algorithm is exponential in worst-case analysis, yet is remarkably successful in practice
- In fact, linear programming can be solved in polynomial time by interior point methods. Recently these have become competitive in practice with simplex.