SOME HINTS FOR PROBLEM SET 2

- The first part of the problem set is calculating the Δv's for each part of the mission. You should not get difficulties with that. It is just like lecture 4. Just remember that you have to assume that each Δv is applied as an **impulse**, which means for the equation giving Δv out of ΔE:

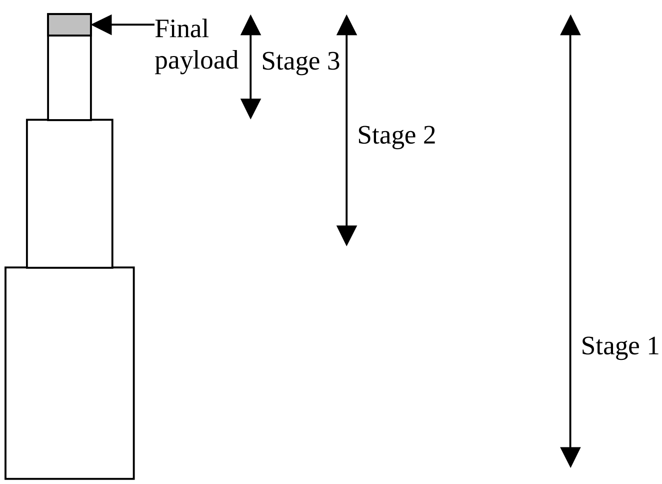
$$\Delta E = \frac{1}{2}\left[2v_i\,\Delta v\underbrace{\cos\phi}_{1(\phi=0)} + (\Delta v)^2\right] - \underbrace{GM\left(\frac{1}{r_f}-\frac{1}{r_i}\right)}_{0}$$

The second term is zero because the spacecraft has no time to move while the Δv is applied.

- Now you have all your Δv's, and you want to calculate the number of stages. The bigger Δv, the more stages you will need (Lecture 2).

Let's work out the example of a three-stage rocket, with, the same exhaust velocity (4,500 m/s), maximum payload, structural and propellant mass fractions for each stage.

When I say 'stage i', that includes (c.f. figure)
  - the classic definition of stage i,
  - the upper stages, which are the payload of this stage.



For each stage:

Payload mass fraction of stage i,

$$m'_{i,pay} = \frac{m_{i,pay}}{m_{i,total}} = 0.1$$

Structural mass fraction

$$m'_{i,struct} = \frac{m_{i,struct}}{m_{i,total}} = 0.1$$

Propellant mass fraction

$$m'_{i,prop} = \frac{m_{i,prop}}{m_{i,total}} = 0.8$$

The total mass of stage i, before it starts burning its propellant being:

$$m_{i,total} = m_{i,pay} + m_{i,struct} + m_{i,prop}$$

Note**: the payload of stage i is, by construction, the total mass of stage i+1**:

$$m_{i,pay} = m_{i+1,total} \qquad (1)$$

The Δv brought by stage i to the final payload is, from the rocket equation:

$$\frac{\Delta v_i}{c} = -\ln\left(\frac{\text{masss of stage i when all its propellant has burnt}}{\text{initial mass of stage i}}\right)$$

where c is the exhaust velocity of the stage-i engine. **The gravity term is zero here because Δv_i is applied as an impulse**.

Plugging in some numbers,

$$\frac{\Delta v_i}{c} = -\ln\left(\frac{m_{i,total} - m_{i,prop}}{m_{i,total}}\right) = -\ln(1-0.8) \cong 1.60$$

$$\Delta v = 1.60 * 4500 = 7{,}200\, m/s$$

For a three-stage rocket, the Δv received by the payload is:

$$\Delta v = \Delta v_1 + \Delta v_2 + \Delta v_3$$

$$\Delta v = 1.60\,(3c) = 21{,}600\, m/s$$

The final payload mass fraction of the rocket, is

$$\frac{m_{payload}}{m_{total}} = \frac{m_{3,pay}}{m_{1,total}} = \frac{m_{3,pay}}{m_{3,total}} \frac{m_{2,pay}}{m_{2,total}} \frac{m_{1,pay}}{m_{1,total}}\text{, using (1)}$$

Hence,

$$\boxed{\frac{m_{payload}}{m_{total}} = m'_{3,pay} m'_{2,pay} m'_{1,pay} = 0.001}$$

If you need a kick of 14,400 m/s only, you can use a two-stage rocket, and the final payload mass fraction will be: 0.1*0.1=0.01

However, if you know you have to give your final payload a kick of, let’s say, 25,000 m/s, then you will have to have 4 stages, the last one being partially used (you do not have to use its maximum propellant mass fraction). Your final payload mass fraction will be smaller than 0.001 because you have added a stage.

In the Problem Set, knowing the mass fractions, the exhaust velocities, you know how much Δv is given by a stage; you also know the required Δv ...