Fluids – Lecture 10 Notes

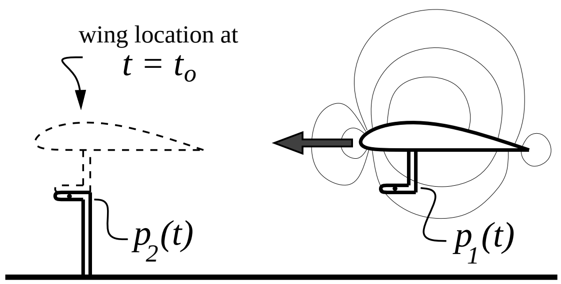
1. Substantial Derivative
2. Recast Governing Equations

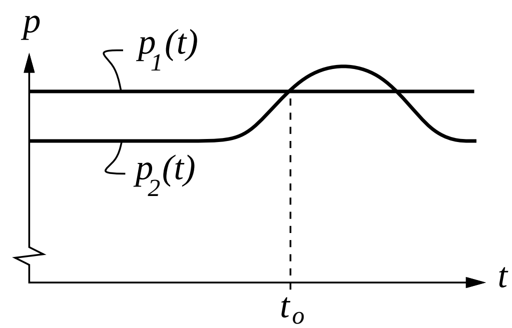
Reading: Anderson 2.9, 2.10

Substantial Derivative

Sensed rates of change

The rate of change reported by a flow sensor clearly depends on the motion of the sensor. For example, the pressure reported by a static-pressure sensor mounted on an airplane in level flight shows zero rate of change. But a ground pressure sensor reports a nonzero rate as the airplane rapidly flies by a few meters overhead. The figure illustrates the situation.





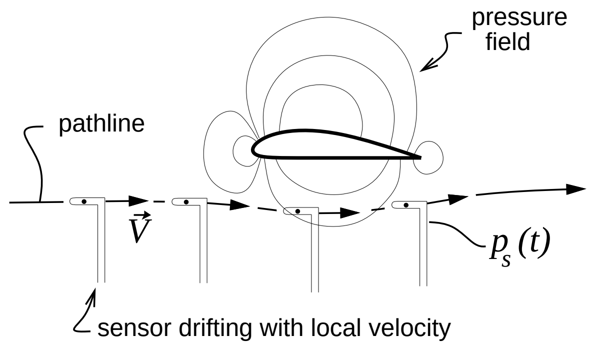
Note that although the two sensors measure the same instantaneous static pressure at the same point (at time $t = t_o$), the measured <u>time rates</u> are different.

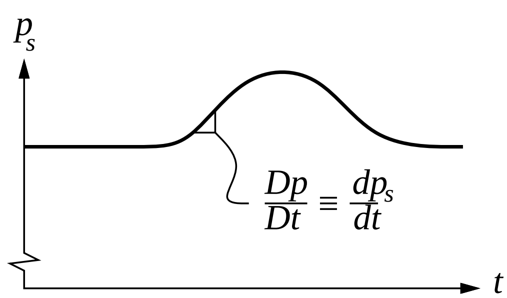
$$p_1(t_o) \; = \; p_2(t_o) \qquad \text{but} \qquad \frac{dp_1}{dt}(t_o) \; \neq \; \frac{dp_2}{dt}(t_o)$$

Drifting sensor

We will now imagine a sensor *drifting with a fluid element*. In effect, the sensor follows the element's pathline coordinates $x_s(t)$, $y_s(t)$, $z_s(t)$, whose time rates of change are just the local flow velocity components

$$\frac{dx_s}{dt} \; = \; u(x_s, y_s, z_s, t) \quad , \qquad \frac{dy_s}{dt} \; = \; v(x_s, y_s, z_s, t) \quad , \qquad \frac{dz_s}{dt} \; = \; w(x_s, y_s, z_s, t)$$





Consider a flow field quantity to be observed by the drifting sensor, such as the static pressure $p(x, y, z, t)$. As the sensor moves through this field, the instantaneous pressure value reported by the sensor is then simply

$$p_s(t) \; = \; p\,(x_s(t), y_s(t), z_s(t), t) \tag{1}$$

This $p_s(t)$ signal is similar to $p_2(t)$ in the example above, but not quite the same, since the p_2 sensor moves in a straight line relative to the wing rather than following a pathline like the p_s sensor.

Substantial derivative definition

The time rate of change of $p_s(t)$ can be computed from (1) using the chain rule.

$$\frac{dp_s}{dt} \;=\; \frac{\partial p}{\partial x}\frac{dx_s}{dt} \;+\; \frac{\partial p}{\partial y}\frac{dy_s}{dt} \;+\; \frac{\partial p}{\partial z}\frac{dz_s}{dt} \;+\; \frac{\partial p}{\partial t}$$

But since dx_s/dt etc. are simply the local fluid velocity components, this rate can be expressed using the flowfield properties alone.

$$\frac{dp_s}{dt} \;=\; \frac{\partial p}{\partial t} \;+\; u\,\frac{\partial p}{\partial x} \;+\; v\,\frac{\partial p}{\partial y} \;+\; w\,\frac{\partial p}{\partial z} \;\equiv\; \frac{Dp}{Dt}$$

The middle expression, conveniently denoted as Dp/Dt in shorthand, is called the *substantial derivative* of p. Note that in order to compute Dp/Dt, we must know not only the $p(x,y,z,t)$ field, but also the velocity component fields $u,v,w(x,y,z,t)$.

Although we used the pressure in this example, the substantial derivative can be computed for <u>any</u> flowfield quantity (density, temperature, even velocity) which is a function of x,y,z,t.

$$\frac{D(\;)}{Dt} \;\equiv\; \frac{\partial(\;)}{\partial t} \;+\; u\,\frac{\partial(\;)}{\partial x} \;+\; v\,\frac{\partial(\;)}{\partial y} \;+\; w\,\frac{\partial(\;)}{\partial z} \;=\; \frac{\partial(\;)}{\partial t} \;+\; \vec{V}\cdot\nabla(\;)$$

The rightmost compact D/Dt definition contains two terms. The first $\partial/\partial t$ term is called the *local derivative*. The second $\vec{V}\cdot\nabla$ term is called the *convective derivative*. In steady flows, $\partial/\partial t = 0$, and only the convective derivative contributes.

Recast Governing Equations

All the governing equations of fluid motion which were derived using control volume concepts can be recast in terms of the substantial derivative. We will employ the following general vector identity

$$\nabla\cdot(a\vec{v}) \;=\; \vec{v}\cdot\nabla a \;+\; a\,\nabla\cdot\vec{v}$$

which is valid for any scalar a and any vector $\vec{v}$.

Continuity equation

Applying the above vector identity to the *divergence form* continuity equation gives

$$\begin{aligned}
\frac{\partial\rho}{\partial t} \;+\; \nabla\cdot\left(\rho\vec{V}\right) \;&=\; 0 \\
\frac{\partial\rho}{\partial t} \;+\; \vec{V}\cdot\nabla\rho \;+\; \rho\,\nabla\cdot\vec{V} \;&=\; 0 \\
\frac{D\rho}{Dt} \;+\; \rho\,\nabla\cdot\vec{V} \;&=\; 0 \qquad (2)
\end{aligned}$$

The final result above is called the *convective form* of the continuity equation. A physical interpretation can be made if it's written as follows.

$$
\begin{aligned}
-\frac{1}{\rho}\frac{D\rho}{Dt} &= \nabla\cdot\vec{V} \\
-\text{fractional density rate} &= \text{velocity divergence} \\
\text{or} \ldots \qquad \text{fractional volume rate} &= \text{velocity divergence}
\end{aligned}
$$

For a fluid element of given mass, the volume must vary as 1/density, which gives the second interpretation above. Both interpretations are illustrated in the left figure below, where the fluid element expands when it flows through a flowfield region where $\nabla\cdot\vec{V} > 0$. In low speed flows and in liquid flows the density is essentially constant, so that $D\rho/Dt = 0$ and by implication $\nabla\cdot\vec{V} = 0$.

Momentum equation

The divergence form of the x-momentum equation is

$$
\frac{\partial(\rho u)}{\partial t} + \nabla\cdot\left(\rho u\vec{V}\right) = -\frac{\partial p}{\partial x} + \rho g_x + (F_x)_{\text{viscous}}
$$

Applying the vector identity again, and also cancelling some terms by use of the continuity equation (2), produces the convective form of the momentum equation. The y- and z-momentum equations are also derived the same way.

$$
\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \rho g_x + (F_x)_{\text{viscous}} \tag{3}
$$

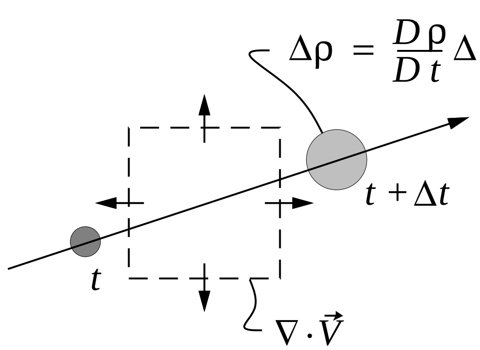
$$
\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \rho g_y + (F_y)_{\text{viscous}} \tag{4}
$$

$$
\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \rho g_z + (F_z)_{\text{viscous}} \tag{5}
$$

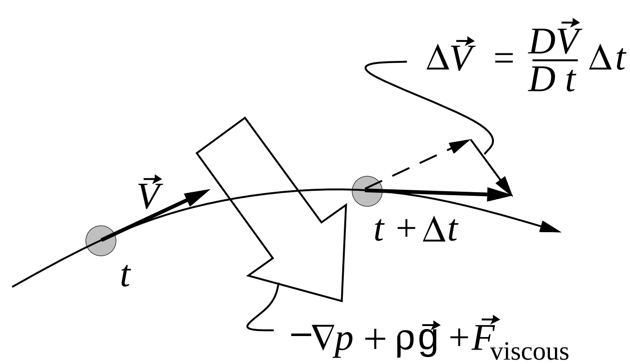
The Du/Dt etc. substantial derivatives are recognized as the acceleration components experienced by a fluid element. This leads to a simple physical interpretation or these equations as Newton's law applied to a fluid element of unit volume.

$$
\text{mass/volume} \times \text{acceleration} = \text{total force/volume}
$$

The element's mass/volume is simply the density ρ, and the total force/volume consists of the buoyancy-like pressure gradient force, the gravity force, and the viscous force.



expanding fluid element



accelerating fluid element