Unit M2.3
(All About) Strain

Readings:
CDL 4.8, 4.9, 4.10
LEARNING OBJECTIVES FOR UNIT M2.3

Through participation in the lectures, recitations, and work associated with Unit M2.3, it is intended that you will be able to........

• .....explain the concept and types of strains and how such is manifested in materials and structures

• .....use the various ways of describing states of strain

• .....describe the relationship between strain and displacement in a body

• .....apply the concept of compatibility to the state of strain
We’ve just talked about how a solid continuum carries load via stress. Now we need to describe how such a continuum deforms. For this, we need to introduce

The Concept of Strain

**Definition**: Strain is the deformation of the continuum at a point or the percentage deformation of an infinitesimal element.

To explore this concept, we need to think about the physical reality of how items deform:

1. Elongation
Consider the change in length, $\Delta l$:

$$\Delta l = l_{\text{deformed}} - l_{\text{undeformed}}$$

(Note: $\Delta l$ can be positive or negative)

Reference this to the original length:

$$\text{Elongation} = E = \frac{l_{\text{deformed}} - l_{\text{undeformed}}}{l_{\text{undeformed}}}$$

--> Now consider the infinitesimal:

(Note: small letters pertain to undeformed; CAPITAL LETTERS to deformed)
The other way in which a body can deform is via.....
2. Shear

This produces an angle change in the body (with no elongations for pure shear)

*Figure M2.3-2*  Illustration of shear deformation of the infinitesimal element

Consider the change in angle:

\[ \Delta \angle = \angle_{\text{deformed}} - \angle_{\text{undeformed}} \]

Would at first make sense.
But, by convention, a reduction in angle is positive shear. So:
\[ \Delta \angle = \angle_{undeformed} - \angle_{deformed} \]
In this case:
\[ \Delta \angle = \left[ \frac{\pi}{2} - \left( \frac{\pi}{2} - \phi \right) \right] = \phi \]

Also note that by keeping this in radians, this is already a nondimensional quantity. [\textbf{Units:} Nondimensional…
\[ \frac{\text{length}}{\text{length}} = \text{"strain";} \quad \mu \text{strain} = 10^{-6} \]

These give us the basic concepts of strain and that there are two types: elongation and shear, but to deal with the full three-dimensional configuration, we need to deal with the….

\textbf{Strain Tensor and Strain Types}

In going from the undeformed (small letters) to the deformed (capital letters) body, we can define a displacement vector, \( \vec{u} \), for any point P.
The overall displacement will have contributions from 4 basic parts:

1. Pure translation (3 directions)
2. Pure rotation (3 planes)
3. Elongation (3 axes/directions)
4. Shear (3 planes)

So we have components of strain.

--> For elongation, need to specify changes of length of three sides of body (so do relative to axes):
For shear, need to specify changes in angles of three sides of body (use planes defined by axes):

\[ \varepsilon_{12} + \varepsilon_{21} = \text{total angle change in } x_1 - x_2 \text{ plane} \]
\[ \varepsilon_{13} + \varepsilon_{31} = \text{total angle change in } x_1 - x_3 \text{ plane} \]
\[ \varepsilon_{23} + \varepsilon_{32} = \text{total angle change in } x_2 - x_3 \text{ plane} \]

Relate to displacement via strain-displacement relations:

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]

Let’s see how we get this....

Formally, the strain tensor is defined by considering the diagonals of the deformed and undeformed elements.
Figure M2.3-4  Position vectors to deformed and undeformed element and the associated diagonals

\[ r = \text{position vector to undeformed element} \]
\[ R = \text{position vector to deformed element} \]
\[ dr = \text{diagonal of undeformed element} \]
\[ dR = \text{diagonal of deformed element} \]

Take the squares of the diagonals:
\[ (ds)^2 = dr \cdot dr \quad (dS)^2 = dR \cdot dR \]
Formal Definition of Strain Tensor

\[(dS)^2 - (ds)^2 = 2\varepsilon_{mn} dx_m dx_n\]

change in magnitude

factor of 2 for angular changes!

\[\varepsilon_{mn} = \text{Strain Tensor}\]

Ref: Bisplinghoff, Mar and Pian, Statics of Deformable Solids, Ch. 5.

But what good does this do us?

This general definition is needed for the most general case with “large strains”, but in many (most engineering) cases we can consider….

Small Strains (vs. Large Strains)

With small deformations in most structures, we can put limits on strains such that:

- changes of length < 10%
- changes of angles < 5%
Good for range of most “engineering materials”

In such cases, higher order terms become negligible and we can equate:

- extensional strain with elongation
- shear strain with angular change

for small strains:
elongation

\[ \varepsilon_{11} \equiv E_{11} = \frac{|PA| - |pa|}{|pa|} \]

where: \( E_{11} = \text{elongation in } x_1 - \text{direction} \)

\[ \varepsilon_{22} \equiv E_{22} = \frac{|PB| - |pb|}{|pb|} \]

and a similar drawing can be made to include \( x_3 \) so that:

\[ \varepsilon_{33} \equiv E_{33} = \frac{|PC| - |pc|}{|pc|} \]

In general:

\[ \text{elongation strain} = \lim_{\text{element length} \to 0} \frac{\text{change in element length}}{\text{element length}} \]

shear:

\[ \varepsilon_{12} \equiv \frac{1}{2} \phi_{12} = \frac{1}{2} \left[ \angle apb - \angle APB \right] \]

where: \( \phi_{12} = \text{angular change in } x_1 - x_2 \text{ plane} \)

And again, drawings to include \( x_3 \) will give:
In general:

\[ \varepsilon_{13} \equiv \frac{1}{2} \phi_{13} = \frac{1}{2} [\angle apc - \angle APC] \]

\[ \varepsilon_{23} \equiv \frac{1}{2} \phi_{23} = \frac{1}{2} [\angle bpc - \angle BPC] \]

Shear strain = 1/2 (angular change)

--> we now have a definition of strain and can deal with the most useful case of “small strain”. But we have not yet defined formally how strain and displacement are related, so we need the:

**Strain - Displacement Relations**

Consider first extensional strains.

We know:

\[ \varepsilon_{11} \equiv \text{elongation in } x_1 \]

\[ \equiv \frac{\ell_{\text{def}} - \ell_{\text{und}}}{\ell_{\text{und}}} \]
Figure M2.3-5  Unit (infinitesimal) element of length $dx_1$

$u_1$ is a field variable $= u_1 (x_1, x_2, x_3)$

$\Rightarrow u_1$ is displacement of left-hand side

$$\left( u_1 + \frac{\partial u_1}{\partial x_1} dx_1 \right)$$

is displacement of right-hand side

way $u_1$ changes with $x_1$

infinitesimal length in $x_1$ - direction

We see:

$$\ell_{\text{undeformed}} = dx_1$$
\[ \ell_{\text{deformed}} = dx_1 + \left( u_1 + \frac{\partial u_1}{\partial x_1} \right) dx_1 - u_1 \]

\[ = dx_1 + \frac{\partial u_1}{\partial x_1} \]

So:

\[ \varepsilon_{11} = \frac{dx_1 + \frac{\partial u_1}{\partial x_1} \, dx_1 - dx_1}{dx_1} \]

\[ \Rightarrow \varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \]

Similarly: (pictures in \( x_2 \) and \( x_3 \) directions)

\[ \varepsilon_{22} = \frac{\partial u_2}{\partial x_2} \]

\[ \varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \]
In general: *extensional* strain is equal to the *rate of change of displacement*

Now consider *shear* strains

We know:

\[ \varepsilon_{12} \equiv \frac{1}{2} \text{ angle change in } x_1 - x_2 \text{ plane } \equiv \frac{1}{2} \phi_{12} \]

\[ = \frac{1}{2} \left\{ \angle_{\text{undef}} - \angle_{\text{def}} \right\} \]

\[ = \frac{1}{2} \left\{ \frac{\pi}{2} - \left( \frac{\pi}{2} - \phi \right) \right\} \]
Figure M2.3-6  Unit (infinitesimal) element $dx_1$ by $dx_2$ in the $x_1 - x_2$ plane

- Using the field variables $u_1(x_1, x_2, x_3)$ and $u_2(x_1, x_2, x_3)$
- Assume small angles such that:  $\tan \theta \approx \theta$
- Start with
  $$\phi = \theta_1 + \theta_2$$
\[ \varphi = \theta_1 + \theta_2 \]

\[ \theta_1 = \left( u_1 + \frac{\partial u_1}{\partial x_2} \, dx_2 \right) - u_1 = \frac{\partial u_1}{\partial x_2} \]

\[ \theta_2 = \left( u_2 + \frac{\partial u_2}{\partial x_1} \, dx_1 \right) - u_2 = \frac{\partial u_2}{\partial x_1} \]

Thus:

\[ \varepsilon_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) = \varepsilon_{21} \]

Recall symmetry of strain tensor

Similarly: (pictures in \( x_1 - x_3 \) and \( x_2 - x_3 \) planes)

\[ \varepsilon_{13} = \varepsilon_{31} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \]
These can be written in general tensor form as:

\[
\varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\]

where:

\[
\mathbf{u} = u_1 \mathbf{i}_1 + u_2 \mathbf{i}_2 + u_3 \mathbf{i}_3
\]

with 6 independent components:

- **extensional**
  - \(\varepsilon_{11}\)
  - \(\varepsilon_{22}\)
  - \(\varepsilon_{33}\)

- **shear**
  - \(\varepsilon_{12} = \varepsilon_{21}\)
  - \(\varepsilon_{13} = \varepsilon_{31}\)
  - \(\varepsilon_{23} = \varepsilon_{32}\)
Note: These relations are developed for small
displacements only. As displacements get
large, must include higher order terms.

It looks like we’re done, but not quite. There is one more concept known as:

Compatibility

One cannot independently describe 3 displacement fields \{u_1 (x_1, x_2, x_3),
u_2 (x_1, x_2, x_3), u_3 (x_1, x_2, x_3)\} by 6 strains

The strains must be related by equations in order for them to be
“compatible”.

Can derive by: (e.g., \(\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}\))

- take second partial of each
\[ \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} = \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2}, \quad \frac{\partial^2 \varepsilon_{22}}{\partial x_2^2} = \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2} \]

\[ \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = \frac{1}{2} \left( \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^3 u_2}{\partial x_1^2 \partial x_2} \right) \]

- substitute first two in latter to get:

\[ \frac{\partial^2 \varepsilon_{11}}{\partial x_2^2} + \frac{\partial^2 \varepsilon_{22}}{\partial x_2^2} - 2 \frac{\partial^2 \varepsilon_{12}}{\partial x_1 \partial x_2} = 0 \]

In general this can be written in tensor form:

\[ \frac{\partial^2 \varepsilon_{nk}}{\partial x_m \partial x_\ell} + \frac{\partial^2 \varepsilon_{ml}}{\partial x_n \partial x_k} - \frac{\partial^2 \varepsilon_{nl}}{\partial x_m \partial x_\ell} - \frac{\partial^2 \varepsilon_{mk}}{\partial x_n \partial x_k} = 0 \]

gives 6 equations (3 conditions)

Are we done? NO…we again need to address…
(More) Strain Notation

Just as in the case of stress, we also need to be familiar with other notations, particularly

--> Engineering Notation

The subscript changes are the same, but there is a fundamental difference with regard to strain

Engineering shear strain = total angle change
Tensorial shear strain = $\frac{1}{2}$ angular change

**BEWARE:** The factor of 2

--> always ask: tensorial or engineering shear strain?

Thus:
In addition, $\gamma$ (gamma) is often used for the shear strains:

\[
\begin{align*}
\gamma_{xy} &= \gamma_{yx} = \varepsilon_{xy} = \varepsilon_{yx} \\
\gamma_{xz} &= \gamma_{zx} = \varepsilon_{xz} = \varepsilon_{zx} \\
\gamma_{yz} &= \gamma_{zy} = \varepsilon_{yz} = \varepsilon_{zy}
\end{align*}
\]

Finally, can also use….

--> Matrix Notation
\[ \varepsilon_{mn} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} \]

symmetric matrix

Finally…

**Deformation/Displacement Notation**

*Figure M2.3-7*  Displacement Notation

- \( p(x_1, x_2, x_3) \),  
  - small \( p \)  
  - (deformed position)

- Capital \( P(x_1, x_2, x_3) \)  
  - (original position)

\[ u_m = p(x_m) - P(x_m) \]
---> Compare notations

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<thead>
<tr>
<th>Tensor</th>
<th>Engineering</th>
<th>Direction in Engineering</th>
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<tbody>
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