The complete solution to systems with inputs

Eytan Modiano

Learning Objectives

- **Analyze linear time-invariant systems with inputs**
  - **Solve for the homogeneous response of the system**
    Natural response without inputs
  - **Solve for the particular solution**
    Identify forced response for different input functions
  - **Obtain the complete solution using initial conditions**
    Complete solution = homogeneous solution + particular solution
  - **Derive the transfer function**

Systems with input

- **In general, systems have inputs**
  - **Applied force in mechanical systems**
  - **Voltage and current sources in circuits**
    - **E.g., battery, power-supply, antenna, scope probe, etc.**

- **Systems also have outputs**
  - **Displays, speakers, voltmeters, etc.**

- **We need to be able to analyze the system response to inputs**
  - **Two methods:**

    Solution to linear constant-coefficients differential equations

    Transfer function methods

Linear constant coefficient differential equations

- **E.g.,** $$\frac{dx(t)}{dt} + 2x(t) = u(t)$$
- **Where x is the state variable and u is the input**
- **The complete solution is of the form:**

$$x(t) = x_p(t) + x_h(t)$$

where x_p is the particular solution (when input spefified)

and x_h is the homogeneous solution to the DE when $u(t) = 0$

$$i.e., \frac{dx(t)}{dt} + 2x(t) = 0$$

- **Thus far we have only considered homogeneous systems**

The particular solution

$$\frac{dx(t)}{dt} + 2x(t) = u(t), \quad u(t) = \begin{cases} 0 & t < 0 \\ e^{3t} & t \geq 0 \end{cases}$$

- **A common method for solving for the particular solution is to try a solution of the same form as the input**
  - **This is called the "forced response"**

- **So try,** $x_p(t) = ae^{3t}$

- **To solve for the constant a, we plug the solution to the original equation**

$$\frac{dx(t)}{dt} + 2x(t) = e^{3t} \Rightarrow 3ae^{3t} + 2ae^{3t} = e^{3t} \Rightarrow a = 1/5$$

$$Particular\ \ solution: \ \ x_p(t) = \frac{e^{3t}}{5}, \ \ t > 0$$

The complete solution

homogeneous solution: $x_h(t) = Be^{st}$

$$Bse^{st} + 2Be^{st} = 0 \Rightarrow Bs + 2B = 0 \Rightarrow B(s+2) = 0 \Rightarrow s = -2$$

$$\Rightarrow x_h(t) = Be^{-2t}$$

$$x(t) = \frac{e^{3t}}{5} + Be^{-2t} \quad , \quad t > 0$$

In order to solve for B, must know initial conditions.

E.g., $\mathrm{x}(0) = 0 \Rightarrow \frac{e^0}{5} + Be^0 = 0 \Rightarrow B = -\frac{1}{5}$

$$x(t) = \frac{1}{5}\left[e^{3t} - e^{-2t}\right], \quad t > 0$$

Key points

- Solution consists of homogeneous and particular solution
  - Homogeneous solution is also called the “natural response”
    - It is the response to zero input
  - The particular solution often takes on the form of the input
    - It is therefore referred to as the “forced response”
- The complete solution requires specification of initial conditions
  - An n^{th} order system would have n initial condition
  - Apply initial conditions to the complete solution in order to obtain the constants
    - The initial conditions are on the complete solution, not just the homogeneous part

Example: RC circuit with inputs

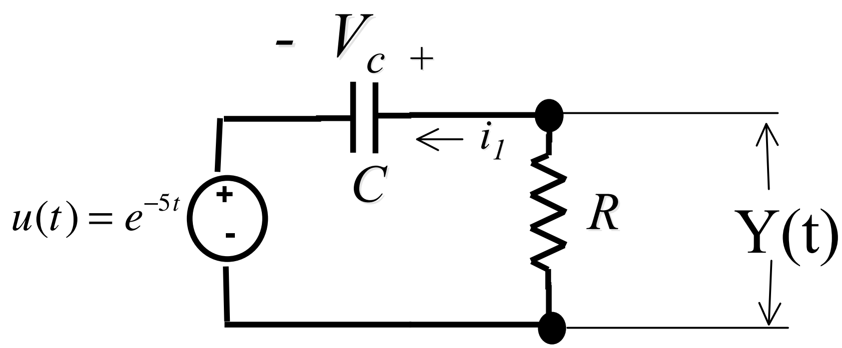
$$\frac{e_1}{R} = -i_1$$

$$\frac{dv_1}{dt} = \frac{i_1}{C} \Rightarrow \frac{dv_1}{dt} = \frac{-e_1}{RC}$$

$$e_1(t) = v_1(t) + u(t)$$

$$\Rightarrow \frac{dv_1}{dt} = \frac{-v_1(t)}{RC} - \frac{u(t)}{RC}$$

$$\text{C=1, R=1} \Rightarrow \frac{dv_1}{dt} = -v_1(t) - u(t)$$



C = 1 F, R = 1 ohm

Homogeneous Solution: $u(t) = 0$

$$Guess\ v_1(t) = ae^{st} \Rightarrow ase^{st} = -ae^{st} \Rightarrow as = -a \Rightarrow s = -1$$

$$\Rightarrow v_H = ae^{-t}$$

The complete solution

Forced Response: $v_F(t) = Be^{-5t}$, $\dot{v}_F = -5Be^{-5t}$

$$\frac{dv_1}{dt} = -v_1(t) - u(t) \Rightarrow -5Be^{-5t} = -Be^{-5t} - e^{-5t}$$

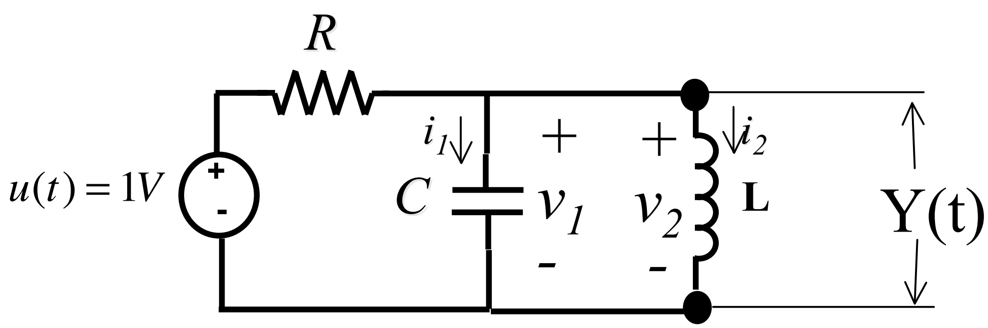
$$\Rightarrow -5B = -B - 1 \Rightarrow B = 1/4 \Rightarrow v_F(t) = \frac{e^{-5t}}{4}$$

$$v_1(t) = v_H(t) + v_F(t) = ae^{-t} + \frac{e^{-5t}}{4}$$

Initial conditions: $v_1(0) = 0v \Rightarrow ae^0 + \frac{e^0}{4} \Rightarrow a + \frac{1}{4} = 0 \Rightarrow a = -\frac{1}{4}$

$$v_1(t) = \frac{-e^{-t} + e^{-5t}}{4}, \quad y(t) = v_1(t) + u(t) = e^{-5t} + \frac{e^{-5t} - e^{-t}}{4}$$

Example: RLC circuit with inputs



Initial conditions: $V_c(0) = 2V$, $i_L(0) = 2A$

Output = Y(t) = Voltage across inductor = $v_2(t)$

Node equation at v_1: $\dfrac{v_1(t) - u(t)}{R} + i_1 + i_2 = 0 \Rightarrow i_1 = \dfrac{v_1}{R} + \dfrac{u}{R} - i_2$

$$\frac{dv_1}{dt} = \frac{i_1}{C} \Rightarrow \frac{dv_1}{dt} = \frac{v_1}{CR} + \frac{u}{CR} - \frac{i_2}{C}$$

$$\frac{di_2}{dt} = \frac{1}{L} v_2 = \frac{v_1}{L}$$

The homogeneous solution (aka: the natural response)

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} -1/RC & -1/C \\ 1/L & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} + \begin{bmatrix} 1/RC \\ 0 \end{bmatrix} u(t)$$

homogeneous solution: take u(t)=0

$$\frac{d}{dt}\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -1/RC & -1/C \\ 1/L & 0 \end{bmatrix}}_{A}\begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$$

Let C=1 F, R=1/2 Ω, L=2 H $\Rightarrow$ A= $\begin{bmatrix} -2 & -1 \\ 1/2 & 0 \end{bmatrix} \Rightarrow SI - A = \begin{bmatrix} s+2 & 1 \\ -1/2 & s \end{bmatrix}$

characteristic equation: $s^2 + 2s + 1/2 = 0$

$$\Rightarrow s_1 = -1 + \frac{1}{\sqrt{2}}, s_1 = -1 - \frac{1}{\sqrt{2}}$$

Natural response: $e^{(-1+1/\sqrt{2})t}$, $e^{(-1-1/\sqrt{2})t}$

The homogeneous solution, continued

Finding the eigen-vectors:

$$s_1 = -1 + \frac{1}{\sqrt{2}} \Rightarrow E^{s_1} = \begin{bmatrix} \frac{-\sqrt{2}}{\sqrt{2}+1} \\ 1 \end{bmatrix}, \quad s_2 = -1 - \frac{1}{\sqrt{2}} \Rightarrow E^{s_2} = \begin{bmatrix} \frac{-\sqrt{2}}{\sqrt{2}-1} \\ 1 \end{bmatrix}$$

$$v_{1n} = a\left(\frac{-\sqrt{2}}{\sqrt{2}+1}\right)e^{(-1+\frac{1}{\sqrt{2}})t} + b\left(\frac{-\sqrt{2}}{\sqrt{2}-1}\right)e^{(-1-\frac{1}{\sqrt{2}})t}, \quad i_{2n} = ae^{(-1+\frac{1}{\sqrt{2}})t} + be^{(-1-\frac{1}{\sqrt{2}})t}$$

Complete solutions: $v_1 = v_{1n} + v_{1f}, \quad i_2 = i_{2n} + i_{2f}$

The particular solution (aka: the forced response)

$u(t) = 1v$

The forced response would be a constant. I.e., $v_{1f} = A,\ i_{2f} = B$

$$\frac{dv_{1f}}{dt} = 0 = \frac{-v_{1f}}{RC} + \frac{u}{RC} - \frac{i_2}{C} = \frac{-A}{RC} + \frac{1}{RC} - \frac{B}{C}$$

$$C = 1F, R = 1/2\Omega, L = 2H \Rightarrow 0 = 2A + 2 - B \Rightarrow 2A + B = 2$$

$$\frac{di_{2f}}{dt} = \frac{v_1}{L} = \frac{A}{L} = 0 \Rightarrow A = 0 \Rightarrow v_{1f} = 0V$$

$$2A + B = 2 \Rightarrow B = 2 \Rightarrow i_{2f} = 2A$$

Does this solution make sense?

The complete solution

Initial conditions: $v_1(0) = 2,\ i_2(0) = 2$

Complete solutions: $v_1 = v_{1n} + v_{1f},\quad i_2 = i_{2n} + i_{2f}$

$$v_1 = a\left(\frac{-\sqrt{2}}{\sqrt{2}+1}\right)e^{(-1+\frac{1}{\sqrt{2}})t} + b\left(\frac{-\sqrt{2}}{\sqrt{2}-1}\right)e^{(-1-\frac{1}{\sqrt{2}})t} + 0$$

$$i_2 = ae^{(-1+\frac{1}{\sqrt{2}})t} + be^{(-1-\frac{1}{\sqrt{2}})t} + 2$$

$$\left.\begin{array}{l} v_1(0) = 2 \Rightarrow a\left(\frac{-\sqrt{2}}{\sqrt{2}+1}\right) + b\left(\frac{-\sqrt{2}}{\sqrt{2}-1}\right) = 2 \\ i_2(0) = 2 \Rightarrow a + b = 2 \Rightarrow a = -b \end{array}\right\} \Rightarrow a = \frac{1}{\sqrt{2}},\quad b = \frac{-1}{\sqrt{2}}$$

$$v_1(t) = \frac{-1}{\sqrt{2}+1}e^{(-1+\frac{1}{\sqrt{2}})t} + \frac{1}{\sqrt{2}-1}e^{(-1-\frac{1}{\sqrt{2}})t}$$

Forced response, $v_{1f} = 0$

Forced response, $i_{2f} = 2$

$$i_2(t) = \frac{1}{\sqrt{2}}e^{(-1+\frac{1}{\sqrt{2}})t} - \frac{1}{\sqrt{2}}e^{(-1-\frac{1}{\sqrt{2}})t} + 2$$