Fluids – Lecture 6 Notes

1. 3-D Vortex Filaments
2. Lifting-Line Theory

Reading: Anderson 5.1

3-D Vortex Filaments

General 3-D vortex

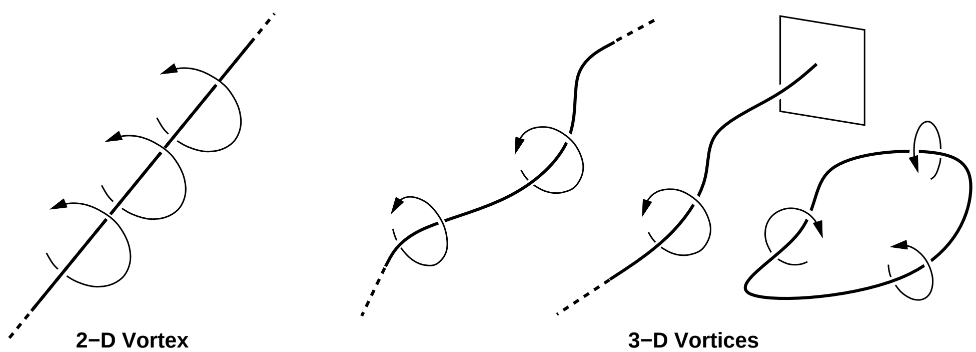
A 2-D vortex, which we have examined previously, can be considered as a 3-D vortex which is straight and extending to $\pm\infty$. Its velocity field is

$$V_\theta = \frac{\Gamma}{2\pi r} \qquad V_r = 0 \qquad V_z = 0 \qquad \text{(2-D vortex)}$$

In contrast, a general 3-D vortex can take any arbitrary shape. However, it is subject to the *Helmholtz Vortex Theorems*:

1) The strength Γ of the vortex is constant all along its length
2) The vortex cannot end inside the fluid. It must either
   a) extend to $\pm\infty$, or
   b) end at a solid boundary, or
   c) form a closed loop.

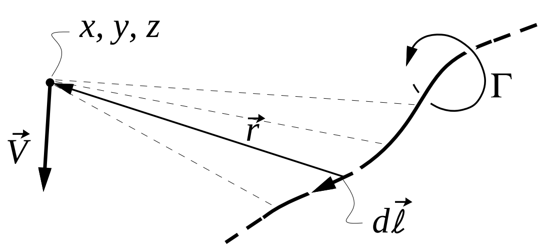
Proofs of these theorems are beyond scope here. However, they are easy to apply in flow modeling situations.



The velocity field of a vortex of general shape is given by the *Biot-Savart Law*.

$$\vec{V}(x,y,z) = \frac{\Gamma}{4\pi}\int_{-\infty}^{+\infty}\frac{d\vec{\ell}\times\vec{r}}{|\vec{r}|^3} \qquad \text{(general 3-D vortex)}$$

The integration is performed along the entire length of the vortex, with $\vec{r}$ extending from the point of integration to the field point x, y, z. The arc length element $d\vec{\ell}$ points along the filament, in the direction of positive Γ by right hand rule.

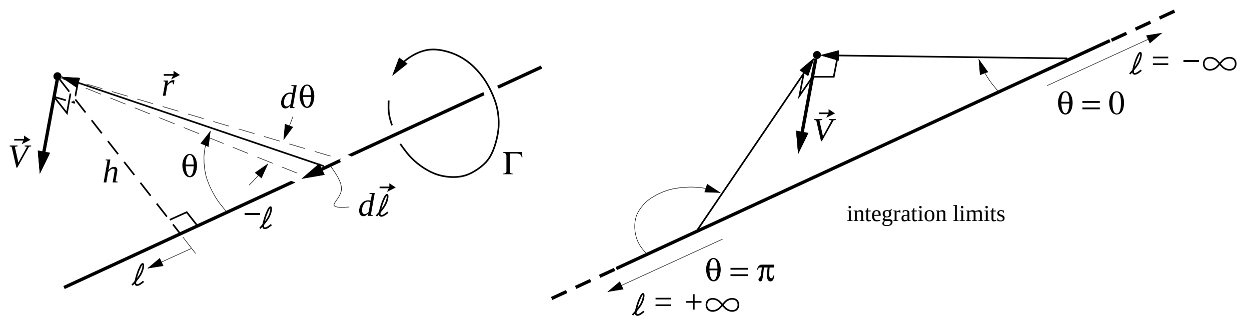


Straight-vortex case

As a check, we can perform the Biot-Savart integral for the case of a straight vortex. Define h as the nearest perpendicular distance of the field point from the vortex line, and θ as the angle between the vortex line and the radius vector $\vec{r}$. We then have

$$
\begin{aligned}
r \equiv |\vec{r}| &= \frac{h}{\sin\theta} \\
\ell &= -\frac{h}{\tan\theta} \\
d\ell &= \frac{h}{\sin^2\theta}\, d\theta \\
d\vec{\ell} \times \vec{r} &= (d\ell \; r \; \sin\theta)\; \hat{\theta}
\end{aligned}
$$

where $\hat{\theta}$ is the unit vector in the tangential direction.



The Biot-Savart integral can now be recast and evaluated as follows.

$$
\vec{V} = \frac{\Gamma}{4\pi h}\,\hat{\theta}\int_0^{\pi}\sin\theta\, d\theta = \frac{\Gamma}{2\pi h}\,\hat{\theta}
$$

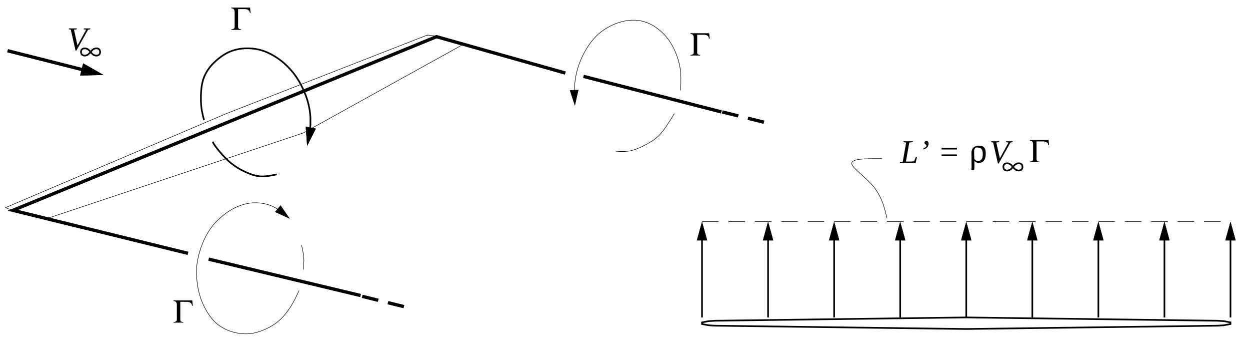
As expected, this recovers the 2-D vortex flowfield $V_\theta = \Gamma/2\pi h$ for this particular case.

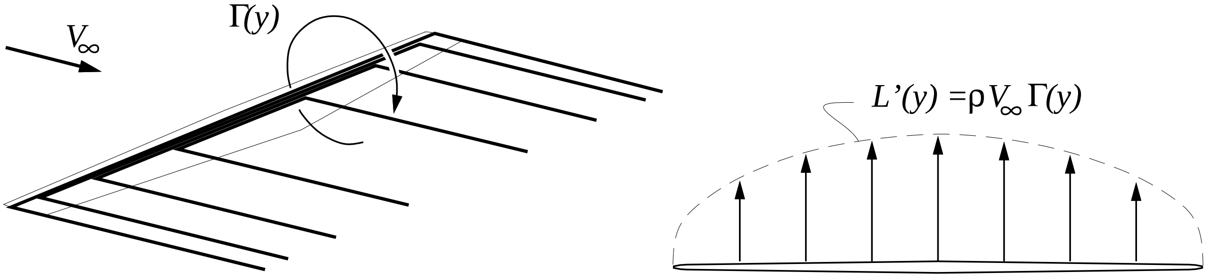
Lifting-Line Theory

Wing vortex model

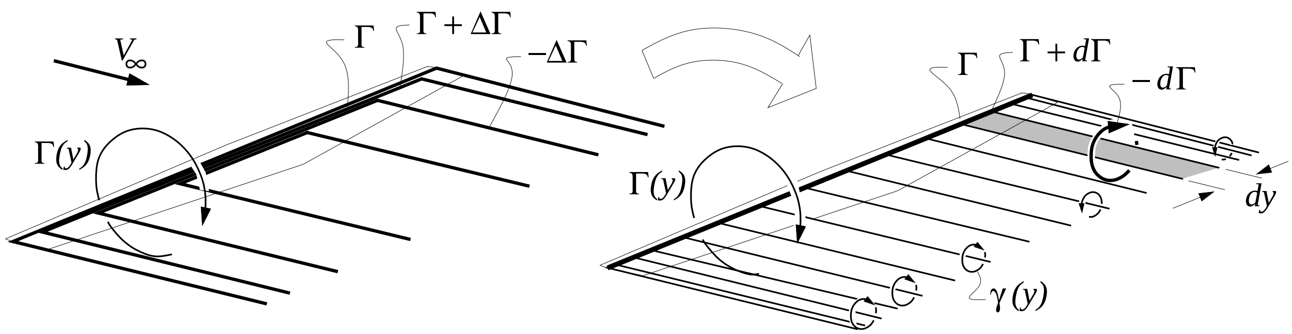
A very simple model for the flowfield about lifting wing is the superposition of a freestream flow and a *horseshoe vortex*. The horseshoe vortex consists of three segments: a *bound vortex* spanning the wing, connected to two *trailing vortices* at each wing tip. As required by Helmholtz's vortex theorems, the circulation Γ is constant along the entire vortex line, and the vortex line extends downstream to infinity. Although this model qualitatively reproduces the observed tip vortices, it is not well suited for accurate prediction of overall wing lift and induced drag. The main deficiency is that its local lift/span $L' = \rho V_\infty \Gamma$ is constant across the span, which is not very realistic. On a real wing, L' always falls gradually to zero at the tips. Another deficiency is that the induced drag predicted by this model is wildly inaccurate, when compared to more refined models or experimental data.

A better flowfield model employs multiple distributed horseshoe vortices as shown in the figure. Each horseshoe vortex has a constant strength along its length and hence obeys Helmholtz's theorem. Spreading the trailing vortices across the span rather than all at the tip allows a nonuniform spanwise circulation $\Gamma(y)$ and corresponding loading $L'(y)$ to be represented.





The figure shows only a few horseshoe vortices on the wing, but one can conceptually subdivide these into more and more vortices of decreasing strength, which in the limit become a *trailing vortex sheet* with strength $\gamma(y)$. The strength of the sheet can be determined by



considering a small change of circulation $d\Gamma$ between spanwise stations y and $y + dy$. By Helmholtz's theorem, the dy-wide sheet strip trailing between those two stations must have a circulation $-d\Gamma$. This then gives the local sheet strength $\gamma(y)$.

$$\gamma \, dy = -d\Gamma$$

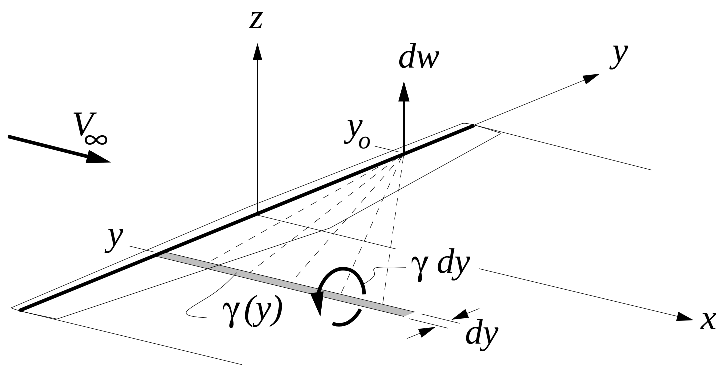
$$\text{or} \qquad \gamma = -\frac{d\Gamma}{dy}$$

Downwash distribution

The trailing vortex sheet will have a downwash distribution $w(y)$ along the span. Consider a dy-wide strip of the sheet at location y, which forms a semi-infinite straight vortex with circulation $\gamma \, dy$. The velocity of this vortex at some other location y_o on the y-axis is

$$dw = \frac{\gamma \, dy}{4\pi(y_o - y)} = -\frac{(d\Gamma/dy) \, dy}{4\pi(y_o - y)}$$

where w is now defined positive up, in the $+z$ direction. The factor of 4π rather than 2π appears because the strip is a semi-infinite vortex, with half the velocity contribution of a doubly-infinite vortex.



Integrating over all the wake strips gives the overall vertical velocity distribution of the whole trailing sheet.

$$w(y_o) \;=\; -\frac{1}{4\pi}\int_{-b/2}^{b/2}\frac{d\Gamma}{dy}\,\frac{dy}{y_o - y}$$

The induced angle distribution is therefore

$$\alpha_i(y_o) \;=\; \frac{-w(y_o)}{V_\infty} \;=\; \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2}\frac{d\Gamma}{dy}\,\frac{dy}{y_o - y}$$

which is defined positive down, as before.

We have obtained an important result, namely a quantitative relation between the circulation distribution $\Gamma(y_o)$, which gives the lift, and the downwash angle distribution $\alpha_i(y_o)$, which will give the lift slope and the induced drag of the wing. The required analysis and calculation method used to obtain the lift and induced drag will be addressed in the subsequent lectures.