

# Fluids – Lecture 19 Notes

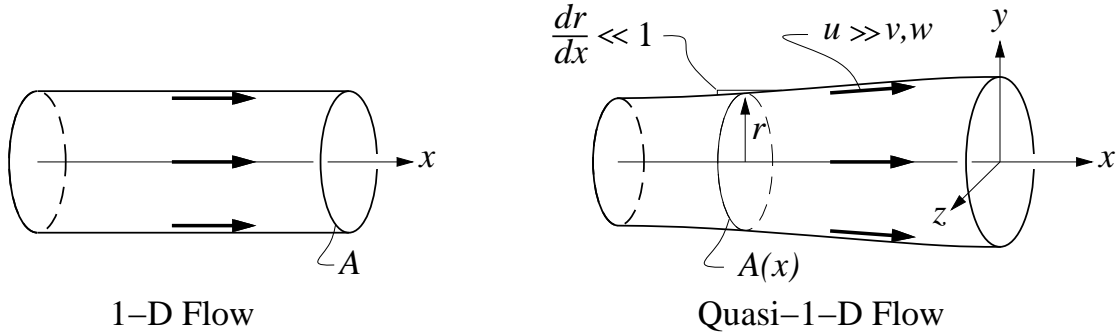
## 1. Compressible Channel Flow

Reading: Anderson 10.1, 10.2

### Compressible Channel Flow

#### Quasi-1-D Flow

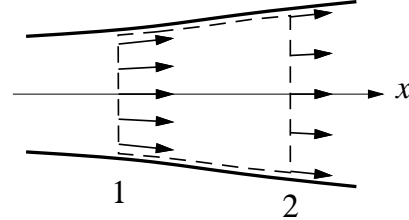
A *quasi-one-dimensional flow* is one in which all variables vary primarily along one direction, say  $x$ . A flow in a duct with slowly-varying area  $A(x)$  is the case of interest here. In practice this means that the slope of the duct walls is small. Also, the  $x$ -velocity component  $u$  dominates the  $y$  and  $z$ -components  $v$  and  $w$ .



#### Governing equations

Application of the integral mass continuity equation to a segment of the duct bounded by any two  $x$  locations gives

$$\begin{aligned} \oint \rho \vec{V} \cdot \hat{n} dA &= 0 \\ -\rho_1 u_1 \iint_1 dA + \rho_2 u_2 \iint_2 dA &= 0 \\ -\rho_1 u_1 A_1 + \rho_2 u_2 A_2 &= 0 \end{aligned}$$



The quasi-1-D approximation is invoked in the second line, with  $u$  and  $\rho$  assumed constant on each cross-sectional area, so they can be taken out of the area integral.

Since stations 1 or 2 can be placed at any arbitrary location  $x$ , we can define the duct *mass flow* which is constant all along the duct, and relates the density, velocity, and area.

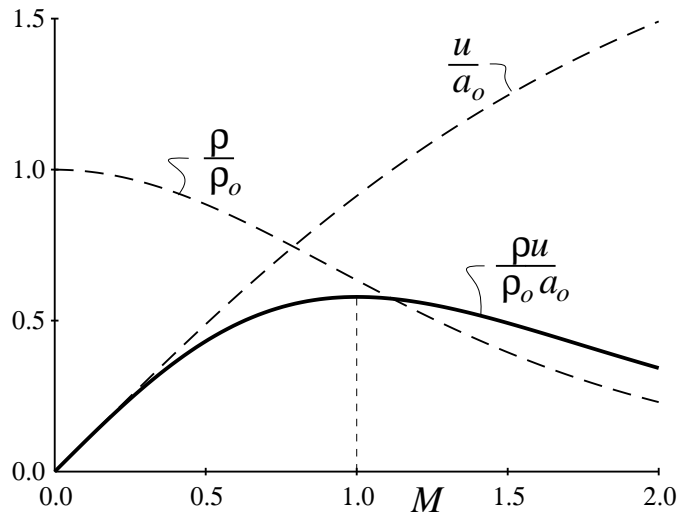
$$\rho(x) u(x) A(x) \equiv \dot{m} = \text{constant} \quad (1)$$

If we assume that the flow in the duct is isentropic, at least piecewise-isentropic between shocks, the stagnation density  $\rho_o$  and stagnation speed of sound  $a_o$  are both constant. This allows the normalized  $\rho$  and  $u$  to be given in terms of the Mach number alone.

$$\frac{\rho}{\rho_o} = \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{1}{\gamma-1}} \quad (2)$$

$$\frac{u}{a_o} = \frac{M a}{a_o} = M \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{1}{2}} \quad (3)$$

$$\frac{\rho u}{\rho_o a_o} = M \left(1 + \frac{\gamma-1}{2} M^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \quad (4)$$



The figure shows these variables, along with the normalized *mass flux*, or  $\rho u$  product, all plotted versus Mach number.

The significance of  $\rho u$  is that it represents the inverse of the duct area, or

$$A \sim \frac{1}{\rho u}$$

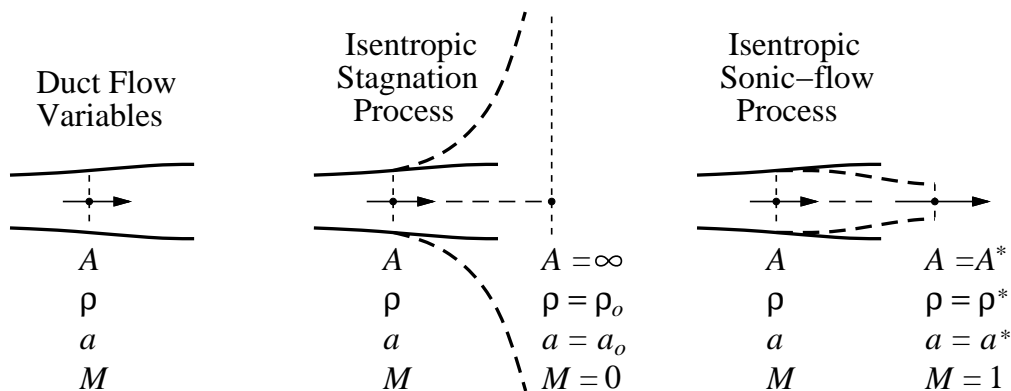
It is evident that the maximum possible mass flux occurs at a location where locally  $M = 1$ . This can be proven by computing

$$\frac{d}{dM} \left( \frac{\rho u}{\rho_0 a_0} \right) = (1 - M^2) \left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{-\frac{\gamma - 3}{2(\gamma - 1)}}$$

which is clearly zero at  $M = 1$ . Therefore, the duct must have a local minimum, or *throat*, wherever  $M = 1$ .

### Sonic conditions

In the development above, the stagnation conditions  $\rho_o$  and  $a_o$  were used to normalize the various quantities. For compressible duct flows, it is very convenient to also define *sonic conditions* which can serve as alternative normalizing quantities. These are defined by a hypothetical process where the flow is sent through a duct of progressively reduced area until  $M = 1$  is reached, shown in the figure along with the familiar stagnation process.



The resulting quantities at the hypothetical sonic throat are denoted by a (\*) superscript. The advantage of the sonic-flow process is that it produces a well-defined *sonic throat area*

$A^*$ , while for the stagnation process  $A$  tends to infinity, and cannot be used for normalization. The ratios between the stagnation and sonic conditions are readily obtained from the usual isentropic relations, with  $M = 1$  plugged in. Numerical values are also given for  $\gamma = 1.4$ .

$$\begin{aligned}\frac{\rho^*}{\rho_o} &= \left(1 + \frac{\gamma-1}{2}\right)^{-\frac{1}{\gamma-1}} = 0.6339 \\ \frac{a^*}{a_o} &= \left(1 + \frac{\gamma-1}{2}\right)^{-\frac{1}{2}} = 0.9129 \\ \frac{p^*}{p_o} &= \left(1 + \frac{\gamma-1}{2}\right)^{-\frac{\gamma}{\gamma-1}} = 0.5283\end{aligned}$$

The sonic flow area  $A^*$  can be obtained from the constant mass flow equation (1). For the sonic-flow process we have

$$\dot{m} = \rho u A = \rho^* u^* A^*$$

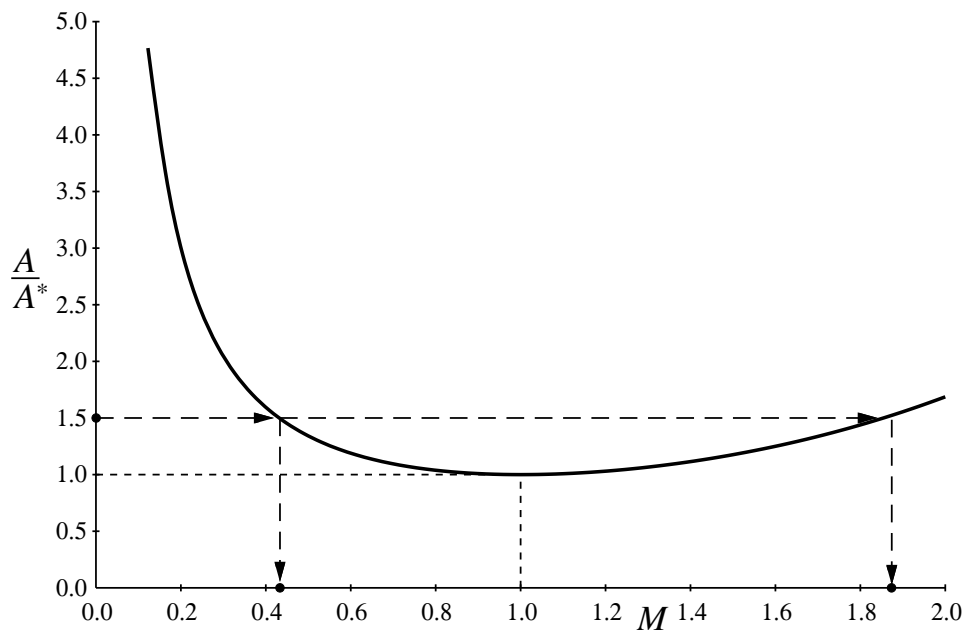
and we also note that  $u^* = a^*$  since  $M = 1$  at the sonic throat. Therefore,

$$\frac{A}{A^*} = \frac{\rho^* a^*}{\rho u} = \frac{\rho^* \rho_o a^* a_o}{\rho_o \rho a_o u}$$

Using the previously-defined expressions produces

$$\frac{A}{A^*} = \frac{1}{M} \left[ \frac{2}{\gamma+1} \left( 1 + \frac{\gamma-1}{2} M^2 \right) \right]^{\frac{\gamma+1}{2(\gamma-1)}} \quad (5)$$

This is the *area-Mach relation*, which is plotted in the figure below for  $\gamma = 1.4$ , and is also available in tabulated form. It uniquely relates the local Mach number to the area ratio  $A/A^*$ , and can be used to “solve” compressible duct flow problems. If the duct geometry  $A(x)$  is given, and  $A^*$  is defined from the known duct mass flow and stagnation quantities, then  $M(x)$  can be determined using the graphical technique shown in the figure, or using the equivalent numerical table.



Once  $M(x)$  is determined, any remaining quantity of interest, such as  $\rho(x)$ ,  $u(x)$ ,  $p(x)$ , etc., can be computed from the isentropic or adiabatic relations such as (2) and (3).

Note that for any given area  $A(x)$ , two solutions are possible for the given mass flow: a subsonic solution with  $M < 1$ , and a supersonic solution with  $M > 1$ . Which solution corresponds to the actual flow depends on whether the flow upstream of that  $x$  location is subsonic or supersonic.

There is also the possibility of shock waves appearing in the duct. This introduces additional complications which will be considered later.